

SCATTERING FOR THE RADIAL 3D CUBIC FOCUSING INHOMOGENEOUS NONLINEAR SCHRÖDINGER EQUATION

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ABSTRACT. The purpose of this work is to study the 3D focusing inhomogeneous nonlinear Schrödinger equation

$$iu_t + \Delta u + |x|^{-b}|u|^2u = 0,$$

where $0 < b < 1/2$. Let Q be the ground state solution of $-Q + \Delta Q + |x|^{-b}|Q|^2Q = 0$ and $s_c = (1+b)/2$. We show that if the radial initial data u_0 belongs to $H^1(\mathbb{R}^3)$ and satisfies $E(u_0)^{s_c} M(u_0)^{1-s_c} < E(Q)^{s_c} M(Q)^{1-s_c}$ and $\|\nabla u_0\|_{L^2}^{s_c} \|u_0\|_{L^2}^{1-s_c} < \|\nabla Q\|_{L^2}^{s_c} \|Q\|_{L^2}^{1-s_c}$, then the corresponding solution is global and scatters in $H^1(\mathbb{R}^3)$. Our proof is based in the ideas introduced by Kenig-Merle [20] in their study of the energy-critical NLS and Holmer-Roudenko [17] for the radial 3D cubic NLS.

1. INTRODUCTION

In this paper, we consider the Cauchy problem, also called the initial value problem (IVP), for the focusing inhomogeneous nonlinear Schrödinger (INLS) equation on \mathbb{R}^3 , that is

$$\begin{cases} i\partial_t u + \Delta u + |x|^{-b}|u|^2u = 0, & t \in \mathbb{R}, x \in \mathbb{R}^3, \\ u(0, x) = u_0(x), \end{cases} \quad (1.1)$$

where $u = u(t, x)$ is a complex-valued function in space-time $\mathbb{R} \times \mathbb{R}^3$ and $0 < b < 1/2$.

Before review some results about the Cauchy problem (1.1), let us recall the critical Sobolev index. For a fixed $\delta > 0$, the rescaled function $u_\delta(t, x) = \delta^{\frac{2-b}{2}} u(\delta^2 t, \delta x)$ is solution of (1.1) if only if $u(t, x)$ is a solution. This scaling property gives rise to a scale-invariant norm. Indeed, computing the homogeneous Sobolev norm of $u_\delta(0, x)$ we get

$$\|u_\delta(0, \cdot)\|_{\dot{H}^s} = \delta^{s - \frac{3}{2} + \frac{2-b}{2}} \|u_0\|_{\dot{H}^s}.$$

Thus, the scale invariant Sobolev space is $H^{s_c}(\mathbb{R}^3)$, where $s_c = \frac{1+b}{2}$ (the critical Sobolev index). Note that, the restriction $0 < b < 1/2$ implies $0 < s_c < 1$ and therefore we are in the mass-supercritical and energy-subcritical case. In addition, we recall that the INLS equation has the following conserved quantities

$$M[u_0] = M[u(t)] = \int_{\mathbb{R}^3} |u(t, x)|^2 dx \quad (1.2)$$

and

$$E[u_0] = E[u(t)] = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u(t, x)|^2 dx - \frac{1}{4} \int_{\mathbb{R}^3} |x|^{-b} |u|^4 dx, \quad (1.3)$$

which are calling Mass and Energy, respectively.

Next, we briefly review recent developments on the well-posedness theory for the general INLS equation

$$\begin{cases} i\partial_t u + \Delta u + |x|^{-b}|u|^\alpha u = 0, & x \in \mathbb{R}^N, \\ u(0, x) = u_0(x). \end{cases} \quad (1.4)$$

Genoud and Stuart [11]-[12], using the abstract theory developed by Cazenave [1] and some sharp Gagliardo-Nirenberg inequalities, showed that (1.4) is well-posed in $H^1(\mathbb{R}^N)$

- locally if $0 < \alpha < 2^*$,
- globally for small initial condition if $\frac{4-2b}{N} < \alpha < \frac{4-2b}{N-2}$,
- globally for any initial condition if $0 < \alpha < \frac{4-2b}{N}$,
- globally if $\alpha = \frac{4-2b}{N}$, assuming $\|u_0\|_{L^2} < \|Q\|_{L^2}$,

where Q is the ground state of the equation $-Q + \Delta Q + |x|^{-b}|Q|^{\frac{4-2b}{N-2}}Q = 0$ and $2^* = \frac{4-2b}{N-2}$, if $N \geq 3$ or $2^* = \infty$, if $N = 1, 2$. Also, Combet and Genoud [3] established the classification of minimal mass blow-up solutions of (1.4) with L^2 critical nonlinearity, that is, $\alpha = \frac{4-2b}{N}$.

Recently, the second author in [15], using the contraction mapping principle based on the Strichartz estimates, proved that the IVP (1.4) is locally well-posed in $H^1(\mathbb{R}^N)$, for $0 < \alpha < 2^*$. Moreover, for $N \geq 2$, $\frac{4-2b}{N} < \alpha < 2^*$ these solutions are global in $H^1(\mathbb{R}^N)$ for small initial data. It worth mentioning that Genoud and Stuart [11]-[12] consider $0 < b < \min\{2, N\}$, and second author in [15] assume $0 < b < \tilde{2}$, where $\tilde{2} = N/3$ if $N = 1, 2, 3$ and $\tilde{2} = 2$ if $N \geq 4$. This new restriction on b is needed to estimate the nonlinear part of the equation in order to use the well known Strichartz estimates associated to the linear flow.

On the other hand, since

$$\|u_\delta\|_{L_x^2} = \delta^{-s_c} \|u\|_{L_x^2}, \quad \|\nabla u_\delta\|_{L_x^2} = \delta^{1-s_c} \|\nabla u\|_{L_x^2} \quad (1.5)$$

and

$$\||x|^{-b}|u_\delta|^4\|_{L_x^1} = \delta^{2(1-s_c)} \||x|^{-b}|u|^4\|_{L_x^1},$$

the following quantities enjoy a scaling invariant property

$$E[u_\delta]^{s_c} M[u_\delta]^{1-s_c} = E[u]^{s_c} M[u]^{1-s_c}, \quad \|\nabla u_\delta\|_{L_x^2}^{s_c} \|u_\delta\|_{L_x^2}^{1-s_c} = \|\nabla u\|_{L_x^2}^{s_c} \|u\|_{L_x^2}^{1-s_c}. \quad (1.6)$$

These quantities were introduced in Holmer-Roudenko [17] in the context of mass-supercritical and energy-subcritical nonlinear Schrödinger equation (NLS), which is equation (1.1) with $b = 0$, and they were used to understand the dichotomy between blowup/global regularity. Indeed, in [17], the authors consider the 3D cubic NLS and proved that if the initial data $u_0 \in H^1(\mathbb{R}^3)$ is radial and satisfy

$$E(u_0)M(u_0) < E(Q)M(Q) \quad (1.7)$$

and

$$\|\nabla u_0\|_{L^2} \|u_0\|_{L^2} < \|\nabla Q\|_{L^2} \|Q\|_{L^2}, \quad (1.8)$$

then the corresponding solution $u(t)$ of the Cauchy problem (1.1) (with $b = 0$) is globally defined and scatters¹ in $H^1(\mathbb{R}^3)$ where Q is the ground state solution of the nonlinear elliptic equation $-Q + \Delta Q + |Q|^2 Q = 0$. The subsequent work Duyckaerts-Holmer-Roudenko [8] has removed the radial assumption on the initial data. In both papers, they used the method of the concentration-compactness and rigidity technique employed by Kenig-Merle [20] in their study of the energy critical NLS. Inspired by these works, we investigate same problem for the IVP (1.1).

Remark 1.1. *The results in Holmer-Roudenko [17] and Duyckaerts-Holmer-Roudenko [8] have been generalized for the general NLS equation (1.4) (with $b = 0$) in the mass-supercritical and energy-subcritical case, by Fang-Xie-Cazenave [9] and Guevara [14]. Moreover, the recent works of Hong [18] and Killip-Murphy-Visan-Zheng [22] also obtained analogous result for the cubic focusing NLS equation perturbed by a potential. It's worth mentioning that global well-posedness and scattering for the mass critical and energy critical NLS has also received a lot of attention in the literature and we refer to Dodson [5]-[6]-[7], Tao-Visan-Zhang [28], Killip-Tao-Visan [23], Killip-Visan-Zhang [25], Colliander-Keel-Staffilani-Takaoka-Tao [2], Ryckman-Visan [27], Visan [29] and Killip-Visan [24] for the results in these directions.*

In a recent work of the first author in [10] showed global well-posedness for the L^2 -supercritical and H^1 -subcritical inhomogeneous nonlinear Schrödinger equation (1.4) under assumptions similar to (1.7)-(1.8). Below we state his result for the 3D cubic INLS, since this is the case we are interested in the present work.

Theorem 1.2. *Let $0 < b < 1$. Suppose that $u(t)$ is the solution of (1.1) with initial data $u_0 \in H^1(\mathbb{R}^3)$ satisfying*

$$E[u_0]^{s_c} M[u_0]^{1-s_c} < E[Q]^{s_c} M[Q]^{1-s_c} \quad (1.9)$$

¹Notice that, in this case the critical Sobolev index is $s_c = 1/2$.

and

$$\|\nabla u_0\|_{L^2}^{s_c} \|u_0\|_{L^2}^{1-s_c} < \|\nabla Q\|_{L^2}^{s_c} \|Q\|_{L^2}^{1-s_c}, \quad (1.10)$$

then $u(t)$ is a global solution in $H^1(\mathbb{R}^3)$. Furthermore, for any $t \in \mathbb{R}$ we have

$$\|\nabla u(t)\|_{L^2}^{s_c} \|u(t)\|_{L^2}^{1-s_c} < \|\nabla Q\|_{L^2}^{s_c} \|Q\|_{L^2}^{1-s_c}, \quad (1.11)$$

where Q is unique positive solution of the elliptic equation

$$-Q + \Delta Q + |x|^{-b}|Q|^2Q = 0. \quad (1.12)$$

Remark 1.3. In [10, Teorema 1.6] the author also considers the case

$$\|\nabla u_0\|_{L^2}^{s_c} \|u_0\|_{L^2}^{1-s_c} > \|\nabla Q\|_{L^2}^{s_c} \|Q\|_{L^2}^{1-s_c}.$$

Indeed assuming the last relation and (1.9) then the solution blows-up in finite time if the initial data u_0 has finite variance, i.e., $|x|u_0 \in L^2(\mathbb{R}^3)$. This is the extension to the INLS model of the result proved by Holmer-Roudenko [16] for the NLS equation.

Our aim in this paper is to show that the global solutions obtained in Theorem 1.2 also scatters (in the radial case) according to the following definition

Definition 1.4. A global solution $u(t)$ to the Cauchy problem (1.1) scatters forward in time in $H^1(\mathbb{R}^3)$, if there exists $\phi^+ \in H^1(\mathbb{R}^3)$ such that

$$\lim_{t \rightarrow +\infty} \|u(t) - U(t)\phi^+\|_{H^1} = 0.$$

Also, we say that $u(t)$ scatters backward in time if there exist $\phi^- \in H^1(\mathbb{R}^3)$ such that

$$\lim_{t \rightarrow -\infty} \|u(t) - U(t)\phi^-\|_{H^1} = 0.$$

Here, $U(t)$ denotes unitary group associated to the linear equation $i\partial_t u + \Delta u = 0$, with initial data u_0 .

The precise statement of our main theorem is the following.

Theorem 1.5. Let $u_0 \in H^1(\mathbb{R}^3)$ be radial and $0 < b < 1/2$. Suppose that (1.9) and (1.10) are satisfied then the solution u of (1.1) is global in $H^1(\mathbb{R}^3)$ and scatters both forward and backward in time.

Remark 1.6. The above theorem extends the result obtained by Holmer-Roudenko [17] to the INLS model. On the other hand, since the solutions of the INLS equation do not enjoy conservation of Momentum, we were not able to use the same ideas introduced by Duyckaerts-Holmer-Roudenko [8] to remove the radial assumption.

The plan of this work is as follows: in the next section we introduce some notations and estimates. In Section 3, we sketch the proof of our main result (Theorem 1.5), assuming all the technical points. In Section 4, we collect some preliminary results about the Cauchy problem (1.1). Next in Section 5, we recall some properties of ground state and show the existence of wave operator. In Section 6, we construct a critical solution denoted by u_c and show some of its properties (the key ingredient in this step is a profile decomposition result related to the linear flow). Finally, Section 7 is devoted to the rigidity theorem.

2. NOTATION AND PRELIMINARIES

Let us start this section by introducing the notation used throughout the paper. We use c to denote various constants that may vary line by line. Given any positive numbers a and b , the notation $a \lesssim b$ means that there exists a positive constant c that $a \leq cb$, with c uniform with respect to the set where a and b vary. Let a set $A \subset \mathbb{R}^3$, $A^C = \mathbb{R}^3 \setminus A$ denotes the complement of A . Given $x, y \in \mathbb{R}^3$, $x \cdot y$ denotes the inner product of x and y in \mathbb{R}^3 .

We use $\|\cdot\|_{L^p}$ to denote the $L^p(\mathbb{R}^3)$ norm with $p \geq 1$. If necessary, we use subscript to inform with variable we are concerned with. The mixed norms in the spaces $L_t^q L_x^r$ and $L_T^q L_x^r$ of $f(x, t)$ are defined, respectively, as

$$\|f\|_{L_t^q L_x^r} = \left(\int_{\mathbb{R}} \|f(t, \cdot)\|_{L_x^r}^q dt \right)^{\frac{1}{q}}$$

and

$$\|f\|_{L_T^q L_x^r} = \left(\int_T^\infty \|f(t, \cdot)\|_{L_x^r}^q dt \right)^{\frac{1}{q}}$$

with the usual modifications when $q = \infty$ or $r = \infty$.

For $s \in \mathbb{R}$, J^s and D^s denote the Bessel and the Riesz potentials of order s , given via Fourier transform by the formulas

$$\widehat{J^s f} = (1 + |y|^2)^{\frac{s}{2}} \widehat{f} \quad \text{and} \quad \widehat{D^s f} = |y|^s \widehat{f},$$

where the Fourier transform of $f(x)$ is given by

$$\widehat{f}(y) = \int_{\mathbb{R}^3} e^{ix \cdot y} f(x) dx.$$

On the other hand, we define the norm of the Sobolev spaces $H^{s,r}(\mathbb{R}^3)$ and $\dot{H}^{s,r}(\mathbb{R}^3)$, respectively, by

$$\|f\|_{H^{s,r}} := \|J^s f\|_{L^r} \quad \text{and} \quad \|f\|_{\dot{H}^{s,r}} := \|D^s f\|_{L^r}.$$

If $r = 2$ we denote $H^{s,2} = H^s$ and $\dot{H}^{s,2} = \dot{H}^s$.

Next, we recall some Strichartz type estimates associated to the linear Schrödinger propagator.

Strichartz type estimates. We say the pair (q, r) is L^2 -admissible or simply admissible par if they satisfy the condition

$$\frac{2}{q} = \frac{3}{2} - \frac{3}{r}, \quad (2.1)$$

where $2 \leq r \leq 6$. We also called the pair \dot{H}^s -admissible if²

$$\frac{2}{q} = \frac{3}{2} - \frac{3}{r} - s, \quad (2.2)$$

where $\frac{6}{3-2s} \leq r \leq 6^-$. Here, a^- is a fixed number slightly smaller than a ($a^- = a - \varepsilon$ with $\varepsilon > 0$ small enough) and, in a similar way, we define a^+ . Finally we say that (q, r) is \dot{H}^{-s} -admissible if

$$\frac{2}{q} = \frac{3}{2} - \frac{3}{r} + s,$$

where $\left(\frac{6}{3-2s}\right)^+ \leq r \leq 6^-$.

Given $s \in \mathbb{R}$, we use the set $\mathcal{A}_s = \{(q, r); (q, r) \text{ is } \dot{H}^s\text{-admissible}\}$ to define the Strichartz norm

$$\|u\|_{S(\dot{H}^s)} = \sup_{(q,r) \in \mathcal{A}_s} \|u\|_{L_t^q L_x^r}.$$

In the same way, the dual Strichartz norm is given by

$$\|u\|_{S'(\dot{H}^{-s})} = \inf_{(q',r') \in \mathcal{A}_{-s}} \|u\|_{L_t^{q'} L_x^{r'}},$$

where (q', r') is such that $\frac{1}{q} + \frac{1}{q'} = 1$ and $\frac{1}{r} + \frac{1}{r'} = 1$ for $(q, r) \in \mathcal{A}_s$.

Note that, if $s = 0$ then \mathcal{A}_0 is the set of all L^2 -admissible pairs. Moreover, if $s = 0$, $S(\dot{H}^0) = S(L^2)$ and $S'(\dot{H}^0) = S'(L^2)$. We write $S(\dot{H}^s)$ or $S'(\dot{H}^{-s})$ if the mixed norm is evaluated over $\mathbb{R} \times \mathbb{R}^3$. To indicate a restriction to a time interval $I \subset (-\infty, \infty)$ and a subset A of \mathbb{R}^3 , we use the notations $S(\dot{H}^s(A); I)$ and $S'(\dot{H}^{-s}(A); I)$.

The next lemmas provide some inequalities that will be useful in our work.

Lemma 2.1. *If $t \neq 0$, $\frac{1}{p} + \frac{1}{p'} = 1$ and $p' \in [1, 2]$, then $U(t) : L^{p'}(\mathbb{R}^3) \rightarrow L^p(\mathbb{R}^3)$ is continuous and*

$$\|U(t)f\|_{L_x^p} \lesssim |t|^{-\frac{3}{2}(\frac{1}{p'} - \frac{1}{p})} \|f\|_{L^{p'}(\mathbb{R}^3)}.$$

Proof. See Linares-Ponce [26, Lemma 4.1]. □

²It worth mentioning that, the pair $(\infty, \frac{6}{3-2s})$ also satisfies the relation (2.2), however, in our work we will not make use of this pair when we estimate the nonlinearity $|x|^{-b}|u|^2u$.

Lemma 2.2. (Sobolev embedding) Let $1 \leq p < +\infty$. If $s \in (0, \frac{3}{2})$ then $H^s(\mathbb{R}^3)$ is continuously embedded in $L^r(\mathbb{R}^3)$ where $s = \frac{3}{p} - \frac{3}{r}$. Moreover,

$$\|f\|_{L^r} \leq c \|D^s f\|_{L^2}. \quad (2.3)$$

Proof. See Linares-Ponce [26, Theorem 3.3]. \square

Remark 2.3. Using Lemma 2.2 we have that $H^s(\mathbb{R}^3)$ is continuously embedded in $L^r(\mathbb{R}^3)$ and

$$\|f\|_{L^r} \leq c \|f\|_{H^s}, \quad (2.4)$$

where $r \in [2, \frac{6}{3-2s}]$.

Next we list the well-known Strichartz estimates we are going to use in this work. We refer the reader to Linares-Ponce [26] and Kato [19] for detailed proofs of what follows (see also Holmer-Roudenko [17] and Guevara [14]).

Lemma 2.4. The following statements hold.

(i) (Linear estimates).

$$\|U(t)f\|_{S(L^2)} \leq c \|f\|_{L^2}, \quad (2.5)$$

$$\|U(t)f\|_{S(\dot{H}^s)} \leq c \|f\|_{\dot{H}^s}. \quad (2.6)$$

(ii) (Inhomogeneous estimates).

$$\left\| \int_{\mathbb{R}} U(t-t')g(.,t')dt' \right\|_{S(L^2)} + \left\| \int_0^t U(t-t')g(.,t')dt' \right\|_{S(L^2)} \leq c \|g\|_{S'(L^2)}, \quad (2.7)$$

$$\left\| \int_0^t U(t-t')g(.,t')dt' \right\|_{S(\dot{H}^s)} \leq c \|g\|_{S'(\dot{H}^{-s})}. \quad (2.8)$$

We end this section with three important remarks.

Remark 2.5. Let $F(x, z) = |x|^{-b}|z|^2z$, and $f(z) = |z|^2z$. The complex derivative of f is $f_z(z) = 2|z|^2$ and $f_{\bar{z}}(z) = z^2$. For $z, w \in \mathbb{C}$, we have

$$f(z) - f(w) = \int_0^1 \left[f_z(w + \theta(z-w))(z-w) + f_{\bar{z}}(w + \theta(z-w))\overline{(z-w)} \right] d\theta.$$

Thus,

$$|F(x, z) - F(x, w)| \lesssim |x|^{-b} (|z|^2 + |w|^2) |z - w|. \quad (2.9)$$

Now we are interested in estimating $\nabla(F(x, z) - F(x, w))$. A simple computation gives

$$\nabla F(x, z) = \nabla(|x|^{-b})f(z) + |x|^{-b}\nabla f(z) \quad (2.10)$$

where $\nabla f(z) = f'(z)\nabla z = f_z(z)\nabla z + f_{\bar{z}}(z)\overline{\nabla z}$.

First we estimate $|\nabla(f(z) - f(w))|$. Note that

$$\nabla(f(z) - f(w)) = f'(z)(\nabla z - \nabla w) + (f'(z) - f'(w))\nabla w. \quad (2.11)$$

So, since

$$|f_z(z) - f_z(w)|, |f_{\bar{z}}(z) - f_{\bar{z}}(w)| \lesssim (|z| + |w|)|z - w|$$

we get, by (2.11)

$$|\nabla(f(z) - f(w))| \lesssim |z|^2|\nabla(z - w)| + (|z| + |w|)|\nabla w||z - w|.$$

Therefore, by (2.10), (2.9) and the last two inequalities we obtain

$$|\nabla(F(x, z) - F(x, w))| \lesssim |x|^{-b-1}(|z|^2 + |w|^2)|z - w| + |x|^{-b}|z|^2|\nabla(z - w)| + M, \quad (2.12)$$

where $M \lesssim |x|^{-b}(|z| + |w|)|\nabla w||z - w|$.

Remark 2.6. Let $B = B(0, 1) = \{x \in \mathbb{R}^3; |x| \leq 1\}$ and $b > 0$. If $x \in B^C$ then $|x|^{-b} < 1$ and so

$$\||x|^{-b}f\|_{L_x^r} \leq \|f\|_{L_x^r(B^C)} + \||x|^{-b}f\|_{L_x^r(B)}.$$

The next remark provides a condition for the integrability of $|x|^{-b}$ on B and B^C .

Remark 2.7. *Note that if $\frac{3}{\gamma} - b > 0$ then $\| |x|^{-b} \|_{L^\gamma(B)} < +\infty$. Indeed*

$$\int_B |x|^{-\gamma b} dx = c \int_0^1 r^{-\gamma b} r^2 dr = c_1 r^{3-\gamma b} \Big|_0^1 < +\infty \text{ if } \frac{3}{\gamma} - b > 0.$$

Similarly, we have that $\| |x|^{-b} \|_{L^\gamma(B^C)}$ is finite if $\frac{3}{\gamma} - b < 0$.

3. SKETCH OF THE PROOF OF THEOREM 1.5

Similarly as in the NLS model, we have the following scattering criteria for global solution in $H^1(\mathbb{R}^3)$ (the proof will be given after Proposition 4.6 below).

Proposition 3.1. (H^1 scattering) *Let $0 < b < 1/2$. If $u(t)$ be a global solution of (1.1) with initial data $u_0 \in H^1(\mathbb{R}^3)$. If $\|u\|_{S(\dot{H}^{s_c})} < +\infty$ and $\sup_{t \in \mathbb{R}} \|u(t)\|_{H_x^1} \leq B$, then $u(t)$ scatters in $H^1(\mathbb{R}^3)$ as $t \rightarrow \pm\infty$.*

Let $u(t)$ be the corresponding H^1 solution for the Cauchy problem (1.1) with radial data $u_0 \in H^1(\mathbb{R}^3)$ satisfying (1.9) and (1.10). We already know by Theorem 1.2 that the solution is globally defined and $\sup_{t \in \mathbb{R}} \|u(t)\|_{H^1} < \infty$. So, in view of Proposition 3.1, our goal is to show that (recalling $s_c = \frac{1+b}{2}$)

$$\|u\|_{S(\dot{H}^{s_c})} < +\infty. \quad (3.1)$$

The technique employed here to achieve the scattering property (3.1) combines the concentration-compactness and rigidity ideas introduced by Kenig-Merle [20]. It is also based on the works of Holmer-Roudenko [17] and Duyckaerts-Holmer-Roudenko [8]. We describe it in the sequel, but first we need some preliminary definitions.

Definition 3.2. *We shall say that $SC(u_0)$ holds if the solution $u(t)$ with initial data $u_0 \in H^1(\mathbb{R}^3)$ is global and (3.1) holds.*

Definition 3.3. *For each $\delta > 0$ define the set A_δ to be the collection of all initial data in $H^1(\mathbb{R}^3)$ satisfying*

$$A_\delta = \{u_0 \in H^1 : E[u_0]^{s_c} M[u_0]^{1-s_c} < \delta \text{ and } \|\nabla u_0\|_{L^2}^{s_c} \|u_0\|_{L^2}^{1-s_c} < \|\nabla Q\|_{L^2}^{s_c} \|Q\|_{L^2}^{1-s_c}\}$$

and define

$$\delta_c = \sup\{\delta > 0 : u_0 \in A_\delta \implies SC(u_0) \text{ holds}\} = \sup_{\delta > 0} B_\delta. \quad (3.2)$$

Note that $B_\delta \neq \emptyset$. In fact, applying the Strichartz estimate (2.6), interpolation and Lemma 5.1 (i) below, we obtain

$$\begin{aligned} \|U(t)u_0\|_{S(\dot{H}^{s_c})} &\leq c \|u_0\|_{\dot{H}^{s_c}} \leq c \|\nabla u_0\|_{L^2}^{s_c} \|u_0\|_{L^2}^{1-s_c} \\ &\leq c \left(\frac{3+b}{s_c} \right)^{\frac{s_c}{2}} E[u_0]^{\frac{s_c}{2}} M[u_0]^{\frac{1-s_c}{2}}. \end{aligned}$$

So if $u_0 \in A_\delta$ then $E[u_0]^{s_c} M[u_0]^{1-s_c} < \left(\frac{s_c}{3+2b} \right)^{s_c} \delta'^2$, which implies $\|U(t)u_0\|_{S(\dot{H}^{s_c})} \leq c\delta'$. Then, by the small data theory (Proposition 4.6 below) we have that $SC(u_0)$ holds for $\delta' > 0$ small enough.

Next, we sketch the proof of Theorem 1.5. If $\delta_c \geq E[Q]^{s_c} M[Q]^{1-s_c}$ then we are done. Assume now, by contradiction, that $\delta_c < E[Q]^{s_c} M[Q]^{1-s_c}$. Therefore, there exists a sequence of radial solutions u_n to (1.1) with H^1 initial data $u_{n,0}$ (rescale all of them to have $\|u_{n,0}\|_{L^2} = 1$ for all n) such that³

$$\|\nabla u_{n,0}\|_{L^2}^{s_c} < \|\nabla Q\|_{L^2}^{s_c} \|Q\|_{L^2}^{1-s_c} \quad (3.3)$$

and

$$E[u_n]^{s_c} \searrow \delta_c \text{ as } n \rightarrow +\infty,$$

³We can rescale $u_{n,0}$ such that $\|u_{n,0}\|_{L^2} = 1$. Indeed, if $u_{n,0}^\lambda(x) = \lambda^{\frac{2-b}{2}} u_{n,0}(\lambda x)$ then by (1.6) we have $E[u_{n,0}^\lambda]^{s_c} M[u_{n,0}^\lambda]^{1-s_c} < E[Q]^{s_c} M[Q]^{1-s_c}$ and $\|\nabla u_{n,0}^\lambda\|_{L^2}^{s_c} \|u_{n,0}^\lambda\|_{L^2}^{1-s_c} < \|\nabla Q\|_{L^2}^{s_c} \|Q\|_{L^2}^{1-s_c}$. Moreover, since $\|u_{n,0}^\lambda\|_{L^2} = \lambda^{-s_c} \|u_{n,0}\|_{L^2}$ by (1.5), setting $\lambda^{s_c} = \|u_{n,0}\|_{L^2}$ we have $\|u_{n,0}^\lambda\|_{L^2} = 1$.

for which $\text{SC}(u_{n,0})$ does not hold for any $n \in \mathbb{R}^3$. However, we already know by Theorem 1.2 that u_n is globally defined. Hence, we must have $\|u_n\|_{S(\dot{H}^{s_c})} = +\infty$. Then using a profile decomposition result (see Proposition 6.1 below) on the sequence $\{u_{n,0}\}_{n \in \mathbb{N}}$ we can construct a critical solution of (1.1), denoted by u_c , that lies exactly at the threshold δ_c , satisfies (3.3) (therefore u_c is globally defined again by Theorem 1.2) and $\|u_c\|_{S(\dot{H}^{s_c})} = +\infty$ (see Proposition 6.4 below). On the other hand, we prove that the critical solution u_c has the property that $K = \{u_c(t) : t \in [0, +\infty)\}$ is precompact in $H^1(\mathbb{R}^3)$ (see Proposition 6.5 below). Finally, the rigidity theorem (Theorem 7.3 below) will imply that such critical solution is identically zero, which contradicts the fact that $\|u_c\|_{S(\dot{H}^{s_c})} = +\infty$.

4. CAUCHY PROBLEM

In this section we show a miscellaneous of results for the Cauchy problem (1.1). These results will be useful in the next sections. We start stating the following two lemmas. To this end, we use the following numbers

$$\widehat{q} = \frac{4(4-\theta)}{6+2b-\theta(1+b)}, \quad \widehat{r} = \frac{6(4-\theta)}{2(3-b)-\theta(2-b)}, \quad (4.1)$$

and

$$\widetilde{a} = \frac{2(4-\theta)}{(7+2b-3\theta)-(2-b)(1-\theta)}, \quad \widehat{a} = \frac{2(4-\theta)}{1-b}. \quad (4.2)$$

It is easy to see that $(\widehat{q}, \widehat{r})$ is L^2 -admissible, $(\widehat{a}, \widehat{r})$ is \dot{H}^{s_c} -admissible and $(\widetilde{a}, \widehat{r})$ is \dot{H}^{-s_c} -admissible.

Lemma 4.1. *Let $0 < b < 1$, then there exist $c > 0$ and a positive number $\theta < 2$ such that*

- (i) $\||x|^{-b}|u|^2v\|_{S'(\dot{H}^{-s_c})} \leq c\|u\|_{L_t^\infty H_x^1}^\theta \|u\|_{S(\dot{H}^{s_c})}^{2-\theta} \|v\|_{S(\dot{H}^{s_c})},$
- (ii) $\||x|^{-b}|u|^2v\|_{S'(L^2)} \leq c\|u\|_{L_t^\infty H_x^1}^\theta \|u\|_{S(\dot{H}^{s_c})}^{2-\theta} \|v\|_{S(L^2)}.$

Proof. (i) We divide the estimate in B and B^C , indeed

$$\||x|^{-b}|u|^2v\|_{S'(\dot{H}^{-s_c})} \leq \||x|^{-b}|u|^2v\|_{S'(\dot{H}^{-s_c}(B))} + \||x|^{-b}|u|^2v\|_{S'(\dot{H}^{-s_c}(B^C))}.$$

We first consider the estimate on B . By the Hölder inequality we deduce

$$\begin{aligned} \||x|^{-b}|u|^2v\|_{L_{\widehat{r}'}(B)} &\leq \||x|^{-b}\|_{L^\gamma(B)} \|u\|_{L_{\theta r_1}^\theta}^\theta \|u\|_{L_{(2-\theta)r_2}^{2-\theta}}^{2-\theta} \|v\|_{L_{\widehat{r}}}, \\ &= \||x|^{-b}\|_{L^\gamma(B)} \|u\|_{L_{\theta r_1}^\theta}^\theta \|u\|_{L_{\widehat{r}}^{2-\theta}}^{2-\theta} \|v\|_{L_{\widehat{r}}}, \end{aligned} \quad (4.3)$$

where

$$\frac{1}{\widehat{r}'} = \frac{1}{\gamma} + \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{\widehat{r}} \quad \text{and} \quad \widehat{r} = (2-\theta)r_2. \quad (4.4)$$

In order to have the norm $\||x|^{-b}\|_{L^\gamma(B)}$ bounded we need $\frac{3}{\gamma} > b$ (see Remark 2.7). In fact, observe that (4.4) implies

$$\frac{3}{\gamma} = 3 - \frac{3(4-\theta)}{\widehat{r}} - \frac{3}{r_1},$$

and from (4.1) it follows that

$$\frac{3}{\gamma} - b = \frac{\theta(2-b)}{2} - \frac{3}{r_1}. \quad (4.5)$$

Choosing $r_1 > 1$ such that $\theta r_1 = 6$ we obtain $\frac{3}{\gamma} - b = \theta(1-b) > 0$ since $b < 1$, that is, $|x|^{-b} \in L^\gamma(B)$. Moreover, using the Sobolev embedding (2.4) (with $s = 1$) and (4.3) we get

$$\||x|^{-b}|u|^2v\|_{L_{\widehat{r}'}(B)} \leq c\|u\|_{H_x^1}^\theta \|u\|_{L_{\widehat{r}}^{2-\theta}}^{2-\theta} \|v\|_{L_{\widehat{r}}}. \quad (4.6)$$

On the other hand, we claim that

$$\||x|^{-b}|u|^2v\|_{L_{\widehat{r}'}(B^C)} \leq c\|u\|_{H_x^1}^\theta \|u\|_{L_{\widehat{r}}^{2-\theta}}^{2-\theta} \|v\|_{L_{\widehat{r}}}. \quad (4.7)$$

Indeed, Arguing in the same way as before we deduce

$$\||x|^{-b}|u|^2v\|_{L_{\widehat{r}'}(B^C)} \leq \||x|^{-b}\|_{L^\gamma(B^C)} \|u\|_{L_{\theta r_1}^\theta}^\theta \|u\|_{L_{\widehat{r}}^{2-\theta}}^{2-\theta} \|v\|_{L_{\widehat{r}}},$$

where the relation (4.5) holds. By Remark 2.7, to show that $\| |x|^{-b} \|_{L^\gamma(B^C)}$ is finite we need to verify that $\frac{3}{\gamma} - b < 0$. Indeed, choosing $r_1 > 1$ such that $\theta r_1 = 2$ and using (4.5) we have $\frac{3}{\gamma} - b = -\frac{\theta(1+b)}{2}$, which is negative. Therefore the Sobolev inequality (2.4) implies (4.7). This completes the proof of the claim.

Now, inequalities (4.6) and (4.7) yield

$$\| |x|^{-b} |u|^2 v \|_{L_{\hat{x}}^{\hat{\gamma}'}} \leq c \|u\|_{H_x^1}^\theta \|u\|_{L_{\hat{x}}^{\hat{\gamma}}}^{2-\theta} \|v\|_{L_{\hat{x}}^{\hat{\gamma}}} \quad (4.8)$$

and the Hölder inequality in the time variable leads to

$$\begin{aligned} \| |x|^{-b} |u|^2 v \|_{L_{\hat{t}}^{\hat{a}'} L_{\hat{x}}^{\hat{\gamma}'}} &\leq c \|u\|_{L_t^\infty H_x^1}^\theta \|u\|_{L_t^{(2-\theta)a_1} L_{\hat{x}}^{\hat{\gamma}}}^{2-\theta} \|v\|_{L_t^{\hat{a}} L_{\hat{x}}^{\hat{\gamma}}} \\ &= c \|u\|_{L_t^\infty H_x^1}^\theta \|u\|_{L_t^{\hat{a}} L_{\hat{x}}^{\hat{\gamma}}}^{2-\theta} \|v\|_{L_t^{\hat{a}} L_{\hat{x}}^{\hat{\gamma}}}, \end{aligned}$$

where

$$\frac{1}{\hat{a}'} = \frac{2-\theta}{\hat{a}} + \frac{1}{\hat{a}}. \quad (4.9)$$

Since \hat{a} and \tilde{a} defined in (4.2) satisfy (4.9) we conclude the proof of item⁴ (i).

(ii) In the previous item we already have (4.8), then applying Hölder's inequality in the time variable we obtain

$$\| |x|^{-b} |u|^2 v \|_{L_{\hat{t}}^{\hat{q}'} L_{\hat{x}}^{\hat{\gamma}'}} \leq c \|u\|_{L_t^\infty H_x^1}^\theta \|u\|_{L_t^{\hat{q}} L_{\hat{x}}^{\hat{\gamma}}}^{2-\theta} \|v\|_{L_t^{\hat{q}} L_{\hat{x}}^{\hat{\gamma}}}, \quad (4.10)$$

since

$$\frac{1}{\hat{q}'} = \frac{2-\theta}{\hat{q}} + \frac{1}{\hat{q}} \quad (4.11)$$

by (4.1) and (4.2). The proof is finished since $(\hat{q}, \hat{\gamma})$ is L^2 -admissible. \square

Remark 4.2. In the perturbation theory we use the following estimate

$$\| |x|^{-b} |u| v w \|_{S'(L^2)} \leq c \|u\|_{L_t^\infty H_x^1}^\theta \|u\|_{S(\dot{H}^{sc})}^{1-\theta} \|v\|_{S(\dot{H}^{sc})} \|w\|_{S(L^2)},$$

where $\theta \in (0, 1)$ is a sufficiently small number. Its proof follows from the ideas of Lemma 4.1 (ii), that is, we can repeat all the computations replacing $|u|^2 v$ by $|u| v w$ or, to be more precise, replacing $|u|^2 v = |u|^\theta |u|^{2-\theta} v$ by $|u| v w = |u|^\theta |u|^{1-\theta} v w$.

Lemma 4.3. Let $0 < b < 1/2$. There exist $c > 0$ and $\theta \in (0, 2)$ sufficiently small such that

$$\| \nabla (|x|^{-b} |u|^2 u) \|_{S'(L^2)} \leq c \|u\|_{L_t^\infty H_x^1}^\theta \|u\|_{S(\dot{H}^{sc})}^{2-\theta} \|\nabla u\|_{S(L^2)}.$$

Proof. Since (2, 6) is L^2 -admissible in 3D and applying the product rule for derivatives we have

$$\begin{aligned} \| \nabla (|x|^{-b} |u|^2 u) \|_{S'(L^2)} &\leq \| |x|^{-b} \nabla (|u|^2 u) \|_{S'(L^2)} + \| \nabla (|x|^{-b}) |u|^2 u \|_{S'(L^2)} \\ &\leq \| |x|^{-b} \nabla (|u|^2 u) \|_{L_t^{\hat{q}'} L_{\hat{x}}^{\hat{\gamma}'}} + \| \nabla (|x|^{-b}) |u|^2 u \|_{L_t^{2'} L_{\hat{x}}^{6'}} \\ &\leq N_1 + N_2. \end{aligned}$$

First, we estimate N_1 (dividing in B and B^C). It follows from Hölder's inequality that

$$\begin{aligned} \| |x|^{-b} \nabla (|u|^2 u) \|_{L_{\hat{x}}^{\hat{\gamma}'}(B)} &\leq \| |x|^{-b} \|_{L^\gamma(B)} \|u\|_{L_x^{\theta r_1}}^\theta \|u\|_{L_x^{(2-\theta)r_2}}^{2-\theta} \|\nabla u\|_{L_{\hat{x}}^{\hat{\gamma}}} \\ &= \| |x|^{-b} \|_{L^\gamma(B)} \|u\|_{L_x^{\theta r_1}}^\theta \|u\|_{L_x^{\hat{\gamma}}}^{2-\theta} \|\nabla u\|_{L_{\hat{x}}^{\hat{\gamma}}}, \end{aligned} \quad (4.12)$$

where

$$\frac{1}{\hat{r}'} = \frac{1}{\gamma} + \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{\hat{r}} \quad \text{and} \quad \hat{r} = (2-\theta)r_2.$$

Notice that the right hand side of (4.12) is the same as the right hand side of (4.3), with $v = \nabla u$. Thus, arguing in the same way as in Lemma 4.1 (i) we obtain

$$\| |x|^{-b} \nabla (|u|^2 u) \|_{L_{\hat{x}}^{\hat{\gamma}'}(B)} \leq c \|u\|_{H_x^1}^\theta \|u\|_{L_{\hat{x}}^{\hat{\gamma}}}^{2-\theta} \|\nabla u\|_{L_{\hat{x}}^{\hat{\gamma}}}.$$

⁴Recall that $(\hat{a}, \hat{\gamma})$ is \dot{H}^{sc} -admissible and $(\tilde{a}, \tilde{\gamma})$ is \dot{H}^{-sc} -admissible.

We also obtain, by Lemma 4.1 (i)

$$\| |x|^{-b} \nabla (|u|^2 u) \|_{L_{\bar{r}'}(B^C)} \leq c \|u\|_{H_x^1}^\theta \|u\|_{L_{\bar{r}}^\infty}^{2-\theta} \|\nabla u\|_{L_x^{\bar{r}}}.$$

Moreover, the Hölder inequality in the time variable leads to (since $\frac{1}{q'} = \frac{2-\theta}{a} + \frac{1}{q}$)

$$\begin{aligned} N_1 = \| |x|^{-b} |u|^2 \nabla u \|_{L_t^{\bar{q}'} L_x^{\bar{r}'}} &\leq c \|u\|_{L_t^\infty H_x^1}^\theta \|u\|_{L_t^{(2-\theta)a_1} L_x^{\bar{r}}}^{2-\theta} \|\nabla u\|_{L_t^{\bar{q}} L_x^{\bar{r}}} \\ &= c \|u\|_{L_t^\infty H_x^1}^\theta \|u\|_{L_t^{\bar{q}} L_x^{\bar{r}}}^{2-\theta} \|\nabla u\|_{L_t^{\bar{q}} L_x^{\bar{r}}}. \end{aligned} \quad (4.13)$$

To estimate N_2 we use the pairs $(\bar{a}, \bar{r}) = \left(8(1-\theta), \frac{12(1-\theta)}{3-2b-\theta(4-2b)}\right)$ \dot{H}^{s_c} -admissible and $(q, r) = \left(\frac{8(1-\theta)}{2-3\theta}, \frac{12(1-\theta)}{4-3\theta}\right)$ L^2 -admissible.⁵ Let $A \subset \mathbb{R}^N$ such that $A = B$ or $A = B^C$. The Hölder inequality and the Sobolev embedding (2.3), with $s = 1$ imply

$$\begin{aligned} \|\nabla (|x|^{-b}) |u|^2 u\|_{L_x^{q'}(A)} &\leq c \| |x|^{-b-1} \|_{L^\gamma(A)} \|u\|_{L_x^{\theta r_1}}^\theta \|u\|_{L_x^{(2-\theta)r_2}}^{2-\theta} \|u\|_{L_x^{r_3}} \\ &\leq c \| |x|^{-b-1} \|_{L^\gamma(A)} \|u\|_{L_x^{\theta r_1}}^\theta \|u\|_{L_x^{\bar{r}}}^{2-\theta} \|\nabla u\|_{L_x^r}, \end{aligned} \quad (4.14)$$

where

$$\frac{1}{6'} = \frac{1}{\gamma} + \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}; \quad 1 = \frac{3}{r} - \frac{3}{r_3}; \quad \bar{r} = (2-\theta)r_2. \quad (4.15)$$

Note that the second equation in (4.15) is valid since $r < 3$. On the other hand, in order to show that $\| |x|^{-b-1} \|_{L^d(A)}$ is bounded, we need $\frac{3}{d} - b - 1 > 0$ when A is the ball B and $\frac{3}{d} - b - 1 < 0$ when $A = B^C$, by Remark 2.7. Indeed, using (4.15) and the values of q, r, \bar{q} and \bar{r} defined above one has

$$\frac{3}{\gamma} - b - 1 = \frac{5}{2} - b - \frac{3}{r_1} - \frac{3(2-\theta)}{\bar{r}} - \frac{3}{r} = \frac{\theta(2-b)}{2} - \frac{3}{r_1}. \quad (4.16)$$

Now choosing r_1 such that

$$\theta r_1 > \frac{6}{2-b} \text{ when } A = B \quad \text{and} \quad \theta r_1 < \frac{6}{2-b} \text{ when } A = B^C,$$

we get $\frac{3}{d} - b - 1 > 0$ when $A = B$ and $\frac{3}{d} - b - 1 < 0$ when $A = B^C$, so $|x|^{-b-1} \in L^d(A)$. In addition, we have by the Sobolev embedding (2.4) (since $2 < \frac{6}{2-b} < 6$) and (4.14)

$$\|\nabla (|x|^{-b}) |u|^2 u\|_{L_x^{q'}(A)} \leq c \|u\|_{H_x^1}^\theta \|u\|_{L_x^{\bar{r}}}^{2-\theta} \|\nabla u\|_{L_x^r}.$$

Finally, by Hölder's inequality in the time variable and the fact that $\frac{1}{2'} = \frac{2-\theta}{a} + \frac{1}{q}$, we conclude

$$N_2 = \|\nabla (|x|^{-b}) |u|^2 u\|_{L_t^{2'} L_x^{q'}} \leq c \|u\|_{L_t^\infty H_x^1}^\theta \|u\|_{L_t^{\bar{a}} L_x^{\bar{r}}}^{2-\theta} \|\nabla u\|_{L_t^q L_x^r}. \quad (4.17)$$

The proof is completed combining (4.13) and (4.17). \square

Remark 4.4. We notice that in Lemma 4.1 and Remark 4.2 we assume $0 < b < 1$. On the other hand, in Lemma 4.3 the required assumption is $0 < b < 1/2$ (see footnote 5). For this reason in our main result, Theorem 1.5, the restriction on b is different than the one in Theorem 1.2.

Remark 4.5. A consequence of the previous lemma is the following estimate

$$\| |x|^{-b-1} |u|^2 v \|_{S'(L^2)} \lesssim \|u\|_{L_t^\infty H_x^1}^\theta \|u\|_{S(\dot{H}^{s_c})}^{2-\theta} \|\nabla v\|_{S(L^2)}.$$

Our first result in this section concerning the IVP (1.1) is the following

⁵Note that $\frac{6}{2-b} = \frac{6}{3-2s_c} < \bar{r} < 6$ (condition of H^s -admissible pair (2.2)). Indeed, it is easy to check that $\bar{r} > \frac{6}{2-b}$. On the other hand, $\bar{r} < 6 \Leftrightarrow \theta(2-2b) < 1-2b$, which is true by the assumption $b < 1/2$ and $\theta > 0$ is a small number. Moreover it is easy to see that $2 < r < 6$, i.e., r satisfies the condition of admissible pair (2.1).

Proposition 4.6. (Small data global theory in H^1) Let $0 < b < 1/2$ and $u_0 \in H^1(\mathbb{R}^3)$. Assume $\|u_0\|_{H^1} \leq A$. There there exists $\delta = \delta(A) > 0$ such that if $\|U(t)u_0\|_{S(\dot{H}^{s_c})} < \delta$, then there exists a unique global solution u of (1.1) such that

$$\|u\|_{S(\dot{H}^{s_c})} \leq 2\|U(t)u_0\|_{S(\dot{H}^{s_c})}$$

and

$$\|u\|_{S(L^2)} + \|\nabla u\|_{S(L^2)} \leq 2c\|u_0\|_{H^1}.$$

Proof. To this end, we use the contraction mapping principle. Define

$$B = \{u : \|u\|_{S(\dot{H}^{s_c})} \leq 2\|U(t)u_0\|_{S(\dot{H}^{s_c})} \text{ and } \|u\|_{S(L^2)} + \|\nabla u\|_{S(L^2)} \leq 2c\|u_0\|_{H^1}\}.$$

We prove that G defined below

$$G(u)(t) = U(t)u_0 + i \int_0^t U(t-t')F(x, u)(t')dt',$$

where $F(x, u) = |x|^{-b}|u|^2u$ is a contraction on B equipped with the metric

$$d(u, v) = \|u - v\|_{S(L^2)} + \|u - v\|_{S(\dot{H}^{s_c})}.$$

Indeed, we deduce by the Strichartz inequalities (2.5), (2.6), (2.7) and (2.8)

$$\|G(u)\|_{S(\dot{H}^{s_c})} \leq \|U(t)u_0\|_{S(\dot{H}^{s_c})} + c\|F\|_{S'(\dot{H}^{-s_c})} \quad (4.18)$$

$$\|G(u)\|_{S(L^2)} \leq c\|u_0\|_{L^2} + c\|F\|_{S'(L^2)} \quad (4.19)$$

$$\|\nabla G(u)\|_{S(L^2)} \leq c\|\nabla u_0\|_{L^2} + c\|\nabla F\|_{S'(L^2)}. \quad (4.20)$$

On the other hand, it follows from Lemmas 4.1 and 4.3 that

$$\|F\|_{S'(\dot{H}^{-s_c})} \leq c\|u\|_{L_t^\infty H_x^1}^\theta \|u\|_{S(\dot{H}^{s_c})}^{2-\theta} \|u\|_{S(\dot{H}^{s_c})}$$

$$\|F\|_{S'(L^2)} \leq c\|u\|_{L_t^\infty H_x^1}^\theta \|u\|_{S(\dot{H}^{s_c})}^{2-\theta} \|u\|_{S(L^2)}$$

$$\|\nabla F\|_{S'(L^2)} \leq c\|u\|_{L_t^\infty H_x^1}^\theta \|u\|_{S(\dot{H}^{s_c})}^{2-\theta} \|\nabla u\|_{S(L^2)}.$$

Combining (4.18)-(4.20) and the last inequalities, we get for $u \in B$

$$\begin{aligned} \|G(u)\|_{S(\dot{H}^{s_c})} &\leq \|U(t)u_0\|_{S(\dot{H}^{s_c})} + c\|u\|_{L_t^\infty H_x^1}^\theta \|u\|_{S(\dot{H}^{s_c})}^{2-\theta} \|u\|_{S(\dot{H}^{s_c})} \\ &\leq \|U(t)u_0\|_{S(\dot{H}^{s_c})} + 8c^{\theta+1}\|u_0\|_{H^1}^\theta \|U(t)u_0\|_{S(\dot{H}^{s_c})}^{3-\theta}. \end{aligned}$$

In addition, setting $X = \|\nabla u\|_{S(L^2)} + \|u\|_{S(L^2)}$ then

$$\begin{aligned} \|G(u)\|_{S(L^2)} + \|\nabla G(u)\|_{S(L^2)} &\leq c\|u_0\|_{H^1} + c\|u\|_{L_t^\infty H_x^1}^\theta \|u\|_{S(\dot{H}^{s_c})}^{2-\theta} X \\ &\leq c\|u_0\|_{H^1} + 16c^{\theta+2}\|u_0\|_{H^1}^{\theta+1} \|U(t)u_0\|_{S(\dot{H}^{s_c})}^{2-\theta}, \end{aligned}$$

where we have used the fact that $X \leq 2^2 c\|u_0\|_{H^1}$ since $u \in B$.

Now if $\|U(t)u_0\|_{S(\dot{H}^{s_c})} < \delta$ with

$$\delta \leq \min \left\{ 2^{-\theta} \sqrt{\frac{1}{16c^{\theta+1}A^\theta}}, 2^{-\theta} \sqrt{\frac{1}{32c^{\theta+1}A^\theta}} \right\}, \quad (4.21)$$

where $A > 0$ is a number such that $\|u_0\|_{H^1} \leq A$, we get

$$\|G(u)\|_{S(\dot{H}^{s_c})} \leq 2\|U(t)u_0\|_{S(\dot{H}^{s_c})}$$

and

$$\|G(u)\|_{S(L^2)} + \|\nabla G(u)\|_{S(L^2)} \leq 2c\|u_0\|_{H^1},$$

that is $G(u) \in B$. The contraction property can be obtained by similar arguments. Therefore, by the Banach Fixed Point Theorem, G has a unique fixed point $u \in B$, which is a global solution of (1.1). \square

We now show Proposition 3.1 (this result gives us the criterion to establish scattering).

Proof of Proposition 3.1. First, we claim that

$$\|u\|_{S(L^2)} + \|\nabla u\|_{S(L^2)} < +\infty. \quad (4.22)$$

Indeed, since $\|u\|_{S(\dot{H}^{s_c})} < +\infty$, given $\delta > 0$ we can decompose $[0, \infty)$ into n many intervals $I_j = [t_j, t_{j+1})$ such that $\|u\|_{S(\dot{H}^{s_c}; I_j)} < \delta$ for all $j = 1, \dots, n$. On the time interval I_j we consider the integral equation

$$u(t) = U(t - t_j)u(t_j) + i \int_{t_j}^{t_{j+1}} U(t - s)(|x|^{-b}|u|^2 u)(s) ds.$$

It follows from the Strichartz estimates (2.5) and (2.7) that

$$\|u\|_{S(L^2; I_j)} \leq c\|u(t_j)\|_{L_x^2} + c\| |x|^{-b}|u|^2 u \|_{S'(L^2; I_j)} \quad (4.23)$$

$$\|\nabla u\|_{S(L^2; I_j)} \leq c\|\nabla u(t_j)\|_{L_x^2} + c\| \nabla(|x|^{-b}|u|^2 u) \|_{S'(L^2; I_j)}. \quad (4.24)$$

From Lemmas 4.1 (ii) and 4.3 we have

$$\| |x|^{-b}|u|^2 u \|_{S'(L^2; I_j)} \leq c\|u\|_{L_{I_j}^\infty H_x^1}^\theta \|u\|_{S(\dot{H}^{s_c}; I_j)}^{2-\theta} \|u\|_{S(L^2; I_j)},$$

$$\| \nabla(|x|^{-b}|u|^2 u) \|_{S'(L^2; I_j)} \leq c\|u\|_{L_{I_j}^\infty H_x^1}^\theta \|u\|_{S(\dot{H}^{s_c}; I_j)}^{2-\theta} \|\nabla u\|_{S(L^2; I_j)}.$$

Thus, using (4.23), (4.24) and the last two estimates we get

$$\|u\|_{S(L^2; I_j)} \leq cB + cB^\theta \delta^{2-\theta} \|u\|_{S(L^2; I_j)}$$

and

$$\|\nabla u\|_{S(L^2; I_j)} \leq cB + cB^{\theta+1} \delta^{2-\theta} + cB^\theta \delta^{2-\theta} \|\nabla u\|_{S(L^2; I_j)}, \quad (4.25)$$

where we have used the assumption $\sup_{t \in \mathbb{R}} \|u(t)\|_{H^1} \leq B$. Taking $\delta > 0$ such that $cB^\theta \delta^{2-\theta} < \frac{1}{2}$ we obtain

$\|u\|_{S(L^2; I_j)} + \|\nabla u\|_{S(L^2; I_j)} \leq cB$, and by summing over the n intervals, we conclude the proof of (4.22).

Returning to the proof of the proposition, let

$$\phi^+ = u_0 + i \int_0^{+\infty} U(-s)|x|^{-b}(|u|^2 u)(s) ds,$$

Note that, $\phi^+ \in H^1(\mathbb{R}^3)$. Indeed, by the same arguments as ones used before we deduce

$$\|\phi^+\|_{L^2} + \|\nabla \phi^+\|_{L^2} \leq c\|u_0\|_{H^1} + c\|u\|_{L_t^\infty H_x^1}^\theta \|u\|_{S(\dot{H}^{s_c})}^{2-\theta} (\|u\|_{S(L^2)} + \|\nabla u\|_{S(L^2)}).$$

Therefore, (4.22) yields $\|\phi\|_{H^1} < +\infty$.

On the other hand, since u is a solution of (1.1) we get

$$u(t) - U(t)\phi^+ = -i \int_t^{+\infty} U(t-s)|x|^{-b}(|u|^2 u)(s) ds.$$

Similarly as before, we have

$$\|u(t) - U(t)\phi\|_{H_x^1} \leq c\|u\|_{L_t^\infty H_x^1}^\theta \|u\|_{S(\dot{H}^{s_c}; [t, \infty))}^{2-\theta} (\|u\|_{S(L^2)} + \|\nabla u\|_{S(L^2)})$$

The proof is completed after using (4.22) and $\|u\|_{S(\dot{H}^{s_c}; [t, \infty))} \rightarrow 0$ as $t \rightarrow +\infty$. \square

Remark 4.7. In the same way we define

$$\phi^- = u_0 + i \int_0^{-\infty} U(-s)|x|^{-b}(|u|^2 u)(s) ds,$$

and using the same argument as before we have $\phi^- \in H^1$ and

$$\|u(t) - U(t)\phi^-\|_{H_x^1} \rightarrow 0 \text{ as } t \rightarrow -\infty.$$

Next, we study the perturbation theory for the IVP (1.1) following the exposition in Killip-Kwon-Shao-Visan [21, Theorem 3.1]. We first obtain a short-time perturbation which can be iterated to obtain a long-time perturbation result.

Proposition 4.8. (Short-time perturbation theory for the INLS) *Let $I \subseteq \mathbb{R}$ be a time interval containing zero and let \tilde{u} defined on $I \times \mathbb{R}^3$ be a solution (in the sense of the appropriated integral equation) to*

$$i\partial_t \tilde{u} + \Delta \tilde{u} + |x|^{-b} |\tilde{u}|^2 \tilde{u} = e,$$

with initial data $\tilde{u}_0 \in H^1(\mathbb{R}^3)$, satisfying

$$\sup_{t \in I} \|\tilde{u}(t)\|_{H_x^1} \leq M \quad \text{and} \quad \|\tilde{u}\|_{S(\dot{H}^{s_c}; I)} \leq \varepsilon, \quad (4.26)$$

for some positive constant M and some small $\varepsilon > 0$.

Let $u_0 \in H^1(\mathbb{R}^3)$ such that

$$\|u_0 - \tilde{u}_0\|_{H^1} \leq M' \quad \text{and} \quad \|U(t)(u_0 - \tilde{u}_0)\|_{S(\dot{H}^{s_c}; I)} \leq \varepsilon, \quad \text{for } M' > 0. \quad (4.27)$$

In addition, assume the following conditions

$$\|e\|_{S'(L^2; I)} + \|\nabla e\|_{S'(L^2; I)} + \|e\|_{S'(\dot{H}^{-s_c}; I)} \leq \varepsilon. \quad (4.28)$$

There exists $\varepsilon_0(M, M') > 0$ such that if $\varepsilon < \varepsilon_0$, then there is a unique solution u to (1.1) on $I \times \mathbb{R}^3$ with initial data u_0 , at the time $t = 0$, satisfying

$$\|u\|_{S(\dot{H}^{s_c}; I)} \lesssim \varepsilon \quad (4.29)$$

and

$$\|u\|_{S(L^2; I)} + \|\nabla u\|_{S(L^2; I)} \lesssim c(M, M'). \quad (4.30)$$

Proof. We use the following claim (we will show it later): there exists $\varepsilon_0 > 0$ sufficiently small such that, if $\|\tilde{u}\|_{S(\dot{H}^{s_c}; I)} \leq \varepsilon_0$ then

$$\|\tilde{u}\|_{S(L^2; I)} \lesssim M \quad \text{and} \quad \|\nabla \tilde{u}\|_{S(L^2; I)} \lesssim M. \quad (4.31)$$

We may assume, without loss of generality, that $0 = \inf I$. Let us first prove the existence of a solution w for the following initial value problem

$$\begin{cases} i\partial_t w + \Delta w + H(x, \tilde{u}, w) + e = 0, \\ w(0, x) = u_0(x) - \tilde{u}_0(x), \end{cases} \quad (4.32)$$

where $H(x, \tilde{u}, w) = |x|^{-b} (|\tilde{u} + w|^2(\tilde{u} + w) - |\tilde{u}|^2 \tilde{u})$.

To this end, let

$$G(w)(t) := U(t)w_0 + i \int_0^t U(t-s)(H(x, \tilde{u}, w) + e)(s)ds \quad (4.33)$$

and define

$$B_{\rho, K} = \{w \in C(I; H^1(\mathbb{R}^3)) : \|w\|_{S(\dot{H}^{s_c}; I)} \leq \rho \text{ and } \|w\|_{S(L^2; I)} + \|\nabla w\|_{S(L^2; I)} \leq K\}.$$

For a suitable choice of the parameters $\rho > 0$ and $K > 0$, we need to show that G in (4.33) defines a contraction on $B_{\rho, K}$. Indeed, applying Strichartz inequalities (2.5), (2.6), (2.7) and (2.8) we have

$$\|G(w)\|_{S(\dot{H}^{s_c}; I)} \lesssim \|U(t)w_0\|_{S(\dot{H}^{s_c}; I)} + \|H(\cdot, \tilde{u}, w)\|_{S'(\dot{H}^{-s_c}; I)} + \|e\|_{S'(\dot{H}^{-s_c}; I)} \quad (4.34)$$

$$\|G(w)\|_{S(L^2; I)} \lesssim \|w_0\|_{L^2} + \|H(\cdot, \tilde{u}, w)\|_{S'(L^2; I)} + \|e\|_{S'(L^2; I)} \quad (4.35)$$

$$\|\nabla G(w)\|_{S(L^2; I)} \lesssim \|\nabla w_0\|_{L^2} + \|\nabla H(\cdot, \tilde{u}, w)\|_{S'(L^2; I)} + \|\nabla e\|_{S'(L^2; I)}. \quad (4.36)$$

On the other hand, since

$$||\tilde{u} + w|^2(\tilde{u} + w) - |\tilde{u}|^2 \tilde{u}| \lesssim |\tilde{u}|^2 |w| + |w|^3 \quad (4.37)$$

by (2.9), we get

$$\|H(\cdot, \tilde{u}, w)\|_{S'(\dot{H}^{-s_c}; I)} \leq \| |x|^{-b} |\tilde{u}|^2 w \|_{S'(\dot{H}^{-s_c}; I)} + \| |x|^{-b} |w|^2 w \|_{S'(\dot{H}^{-s_c}; I)},$$

which implies using Lemma 4.1 (i) that

$$\|H(\cdot, \tilde{u}, w)\|_{S'(\dot{H}^{-s_c}; I)} \lesssim \left(\|\tilde{u}\|_{L_t^\infty H_x^1}^\theta \|\tilde{u}\|_{S(\dot{H}^{s_c}; I)}^{2-\theta} + \|w\|_{L_t^\infty H_x^1}^\theta \|w\|_{S(\dot{H}^{s_c}; I)}^{2-\theta} \right) \|w\|_{S(\dot{H}^{s_c}; I)}. \quad (4.38)$$

The same argument and Lemma 4.1 (ii) also yield

$$\|H(\cdot, \tilde{u}, w)\|_{S'(L^2; I)} \lesssim \left(\|\tilde{u}\|_{L_t^\infty H_x^1}^\theta \|\tilde{u}\|_{S(\dot{H}^{s_c}; I)}^{2-\theta} + \|w\|_{L_t^\infty H_x^1}^\theta \|w\|_{S(\dot{H}^{s_c}; I)}^{2-\theta} \right) \|w\|_{S(L^2; I)}. \quad (4.39)$$

Now, we estimate $\|\nabla H(\cdot, \tilde{u}, w)\|_{S'(L^2; I)}$. It follows from (2.12) and (4.37) that

$$|\nabla H(x, \tilde{u}, w)| \lesssim |x|^{-b-1}(|\tilde{u}|^2 + |w|^2)|w| + |x|^{-b}(|\tilde{u}|^2 + |w|^2)|\nabla w| + E,$$

where $E \lesssim |x|^{-b}(|\tilde{u}| + |w|)|w||\nabla \tilde{u}|$. Thus, Lemma 4.1 (ii), Remark 4.5 and Remark 4.2 lead to

$$\begin{aligned} \|\nabla H(\cdot, \tilde{u}, w)\|_{S'(L^2; I)} &\lesssim \left(\|\tilde{u}\|_{L_t^\infty H_x^1}^\theta \|\tilde{u}\|_{S(\dot{H}^{s_c}; I)}^{2-\theta} + \|w\|_{L_t^\infty H_x^1}^\theta \|w\|_{S(\dot{H}^{s_c}; I)}^{2-\theta} \right) \|\nabla w\|_{S(L^2; I)} \\ &\quad + \left(\|\tilde{u}\|_{L_t^\infty H_x^1}^\theta \|\tilde{u}\|_{S(\dot{H}^{s_c}; I)}^{1-\theta} + \|w\|_{L_t^\infty H_x^1}^\theta \|w\|_{S(\dot{H}^{s_c}; I)}^{1-\theta} \right) \|w\|_{S(\dot{H}^{s_c}; I)} \|\nabla \tilde{u}\|_{S(L^2; I)} \end{aligned} \quad (4.40)$$

Hence, combining (4.38), (4.39) and if $u \in B(\rho, K)$, we have

$$\|H(\cdot, \tilde{u}, w)\|_{S'(\dot{H}^{-s_c}; I)} \lesssim (M^\theta \varepsilon^{2-\theta} + K^\theta \rho^{2-\theta}) \rho \quad (4.41)$$

$$\|H(\cdot, \tilde{u}, w)\|_{S'(L^2; I)} \lesssim (M^\theta \varepsilon^{2-\theta} + K^\theta \rho^{2-\theta}) K. \quad (4.42)$$

Furthermore, (4.40) and (4.31) imply

$$\|\nabla H(\cdot, \tilde{u}, w)\|_{S'(L^2; I)} \lesssim (M^\theta \varepsilon^{2-\theta} + K^\theta \rho^{2-\theta}) K + (M^\theta \varepsilon^{1-\theta} + K^\theta \rho^{1-\theta}) \rho M. \quad (4.43)$$

Therefore, we deduce by (4.34)-(4.35) together with (4.41)-(4.42) that

$$\|G(w)\|_{S(\dot{H}^{s_c}; I)} \leq c\varepsilon + cA\rho$$

$$\|G(w)\|_{S(L^2; I)} \leq cM' + c\varepsilon + cAK,$$

where we also used the hypothesis (4.27)-(4.28) and $A = M^\theta \varepsilon^{2-\theta} + K^\theta \rho^{2-\theta}$. We also have, using (4.36), (4.43)

$$\|\nabla G(w)\|_{S(L^2; I)} \leq cM' + c\varepsilon + cAK + cB\rho M,$$

where $B = M^\theta \varepsilon^{1-\theta} + K^\theta \rho^{1-\theta}$.

Choosing $\rho = 2c\varepsilon$, $K = 3cM'$ and ε_0 sufficiently small such that

$$cA < \frac{1}{3} \quad \text{and} \quad c(\varepsilon + B\rho M + K^\theta \rho^{2-\theta} M) < \frac{K}{3},$$

we obtain

$$\|G(w)\|_{S(\dot{H}^{s_c}; I)} \leq \rho \quad \text{and} \quad \|G(w)\|_{S(L^2; I)} + \|\nabla G(w)\|_{S(L^2; I)} \leq K.$$

The above calculations establish that G is well defined on $B(\rho, K)$. The contraction property can be obtained by similar arguments. Hence, by the Banach Fixed Point Theorem we obtain a unique solution w on $I \times \mathbb{R}^N$ such that

$$\|w\|_{S(\dot{H}^{s_c}; I)} \lesssim \varepsilon \quad \text{and} \quad \|w\|_{S(L^2; I)} + \|w\|_{S(L^2; I)} \lesssim M'.$$

Finally, it is easy to see that $u = \tilde{u} + w$ is a solution to (1.1) satisfying (4.29) and (4.30).

To complete the proof we now show (4.31). Indeed, we first show that

$$\|\nabla \tilde{u}\|_{S(L^2; I)} \lesssim M. \quad (4.44)$$

Using the same arguments as before, we have

$$\|\nabla \tilde{u}\|_{S(L^2; I)} \lesssim \|\nabla \tilde{u}_0\|_{L^2} + \left\| \nabla(|x|^{-b}|\tilde{u}|^2\tilde{u}) \right\|_{S'(L^2; I)} + \|\nabla e\|_{S'(L^2; I)}.$$

Lemma 4.3 implies

$$\begin{aligned} \|\nabla \tilde{u}\|_{S(L^2; I)} &\lesssim M + \|\tilde{u}\|_{L_t^\infty H_x^1}^\theta \|\tilde{u}\|_{S(\dot{H}^{s_c}; I)}^{2-\theta} \|\nabla \tilde{u}\|_{S(L^2; I)} + \varepsilon \\ &\lesssim M + \varepsilon + M^\theta \varepsilon^{2-\theta} \|\nabla \tilde{u}\|_{S(L^2; I)}. \end{aligned}$$

Therefore, choosing ε_0 sufficiently small the linear term $M^\theta \varepsilon_0^{2-\theta} \|\nabla \tilde{u}\|_{S(L^2;I)}$ may be absorbed by the left-hand term and we conclude the proof of (4.44). Similar estimates also imply $\|\tilde{u}\|_{S(L^2;I)} \lesssim M$. \square

Remark 4.9. From Proposition 4.8, we also have the following estimates:

$$\|H(\cdot, \tilde{u}, w)\|_{S'(\dot{H}^{-s_c};I)} \leq C(M, M')\varepsilon \quad (4.45)$$

and

$$\|H(\cdot, \tilde{u}, w)\|_{S'(L^2;I)} + \|\nabla H(\cdot, \tilde{u}, w)\|_{S'(L^2;I)} \leq C(M, M')\varepsilon^{2-\theta}, \quad (4.46)$$

with $\theta > 0$ small enough. Indeed, the relations (4.41), (4.42) and (4.43) imply

$$\|H(\cdot, \tilde{u}, w)\|_{S'(\dot{H}^{-s_c};I)} \lesssim (M^\theta \varepsilon^{2-\theta} + K^\theta \rho^{2-\theta}) \rho,$$

$$\|H(\cdot, \tilde{u}, w)\|_{S'(L^2;I)} \lesssim (M^\theta \varepsilon^{2-\theta} + K^\theta \rho^{2-\theta}) K$$

and

$$\|\nabla H(\cdot, \tilde{u}, w)\|_{S'(L^2;I)} \lesssim (M^\theta \varepsilon^{2-\theta} + K^\theta \rho^{2-\theta}) K + (M^\theta \varepsilon^{1-\theta} + K^\theta \rho^{1-\theta}) \rho M.$$

Therefore, the choice $\rho = 2c\varepsilon$ and $K = 3cM'$ in Proposition 4.8 yield (4.45) and (4.46).

In the sequel, we prove the long-time perturbation result.

Proposition 4.10. (Long-time perturbation theory for the INLS) Let $I \subseteq \mathbb{R}$ be a time interval containing zero and let \tilde{u} defined on $I \times \mathbb{R}^3$ be a solution (in the sense of the appropriated integral equation) to

$$i\partial_t \tilde{u} + \Delta \tilde{u} + |x|^{-b} |\tilde{u}|^2 \tilde{u} = e,$$

with initial data $\tilde{u}_0 \in H^1(\mathbb{R}^3)$, satisfying

$$\sup_{t \in I} \|\tilde{u}\|_{H_x^1} \leq M \quad \text{and} \quad \|\tilde{u}\|_{S(\dot{H}^{s_c};I)} \leq L, \quad (4.47)$$

for some positive constants M, L .

Let $u_0 \in H^1(\mathbb{R}^3)$ such that

$$\|u_0 - \tilde{u}_0\|_{H^1} \leq M' \quad \text{and} \quad \|U(t)(u_0 - \tilde{u}_0)\|_{S(\dot{H}^{s_c};I)} \leq \varepsilon, \quad (4.48)$$

for some positive constant M' and some $0 < \varepsilon < \varepsilon_1 = \varepsilon_1(M, M', L)$. Moreover, assume also the following conditions

$$\|e\|_{S'(L^2;I)} + \|\nabla e\|_{S'(L^2;I)} + \|e\|_{S'(\dot{H}^{-s_c};I)} \leq \varepsilon.$$

Then, there exists a unique solution u to (1.1) on $I \times \mathbb{R}^3$ with initial data u_0 at the time $t = 0$ satisfying

$$\|u - \tilde{u}\|_{S(\dot{H}^{s_c};I)} \leq C(M, M', L)\varepsilon \quad \text{and} \quad (4.49)$$

$$\|u\|_{S(\dot{H}^{s_c};I)} + \|u\|_{S(L^2;I)} + \|\nabla u\|_{S(L^2;I)} \leq C(M, M', L). \quad (4.50)$$

Proof. First observe that since $\|\tilde{u}\|_{S(\dot{H}^{s_c};I)} \leq L$, given⁶ $\varepsilon < \varepsilon_0(M, 2M')$ we can partition I into $n = n(L, \varepsilon)$ intervals $I_j = [t_j, t_{j+1})$ such that for each j , the quantity $\|\tilde{u}\|_{S(\dot{H}^{s_c};I_j)} \leq \varepsilon$. Note that M' is being replaced by $2M'$, as the H^1 -norm of the difference of two different initial data may increase in each iteration.

Again, we may assume, without loss of generality, that $0 = \inf I$. Let w be defined by $u = \tilde{u} + w$, then w solves IVP (4.32) with initial time t_j . Thus, the integral equation in the interval $I_j = [t_j, t_{j+1})$ reads as follows

$$w(t) = U(t - t_j)w(t_j) + i \int_{t_j}^t U(t - s)(H(x, \tilde{u}, w) + e)(s)ds,$$

where $H(x, \tilde{u}, w) = |x|^{-b} (|\tilde{u} + w|^2(\tilde{u} + w) - |\tilde{u}|^2 \tilde{u})$.

Thus, choosing ε_1 sufficiently small (depending on n, M , and M'), we may apply Proposition 4.8 (Short-time perturbation theory) to obtain for each $0 \leq j < n$ and all $\varepsilon < \varepsilon_1$,

$$\|u - \tilde{u}\|_{S(\dot{H}^{s_c};I_j)} \leq C(M, M', j)\varepsilon \quad (4.51)$$

⁶ ε_0 is given by the previous result and ε to be determined later.

and

$$\|w\|_{S(\dot{H}^{s_c}; I_j)} + \|w\|_{S'(L^2; I_j)} + \|\nabla w\|_{S'(L^2; I_j)} \leq C(M, M', j) \quad (4.52)$$

provided we can show

$$\|U(t - t_j)(u(t_j) - \tilde{u}(t_j))\|_{S(\dot{H}^{s_c}; I_j)} \leq C(M, M', j)\varepsilon \leq \varepsilon_0 \quad (4.53)$$

and

$$\|u(t_j) - \tilde{u}(t_j)\|_{H_x^1} \leq 2M', \quad (4.54)$$

For each $0 \leq j < n$.

Indeed, by the Strichartz estimates (2.6) and (2.8), we have

$$\begin{aligned} \|U(t - t_j)w(t_j)\|_{S(\dot{H}^{s_c}; I_j)} &\lesssim \|U(t)w_0\|_{S(\dot{H}^{s_c}; I)} + \|H(\cdot, \tilde{u}, w)\|_{S'(\dot{H}^{-s_c}; [0, t_j])} \\ &\quad + \|e\|_{S'(\dot{H}^{-s_c}; I)}, \end{aligned}$$

which implies by (4.45) that

$$\|U(t - t_j)(u(t_j) - \tilde{u}(t_j))\|_{S(\dot{H}^{s_c}; I_j)} \lesssim \varepsilon + \sum_{k=0}^{j-1} C(k, M, M')\varepsilon.$$

Similarly, it follows from Strichartz estimates (2.5), (2.7) and (4.46) that

$$\begin{aligned} \|u(t_j) - \tilde{u}(t_j)\|_{H_x^1} &\lesssim \|u_0 - \tilde{u}_0\|_{H^1} + \|e\|_{S'(L^2; I)} + \|\nabla e\|_{S'(L^2; I)} \\ &\quad + \|H(\cdot, \tilde{u}, w)\|_{S'(L^2; [0, t_j])} + \|\nabla H(\cdot, \tilde{u}, w)\|_{S'(L^2; [0, t_j])} \\ &\lesssim M' + \varepsilon + \sum_{k=0}^{j-1} C(k, M, M')\varepsilon^{2-\theta}. \end{aligned}$$

Taking $\varepsilon_1 = \varepsilon(n, M, M')$ sufficiently small, we see that (4.53) and (4.54) hold and so, it implies (4.51) and (4.52).

Finally, summing this over all subintervals I_j we obtain

$$\|u - \tilde{u}\|_{S(\dot{H}^{s_c}; I)} \leq C(M, M', L)\varepsilon$$

and

$$\|w\|_{S(\dot{H}^{s_c}; I)} + \|w\|_{S'(L^2; I)} + \|\nabla w\|_{S'(L^2; I)} \leq C(M, M', L).$$

This completes the proof. \square

5. PROPERTIES OF THE GROUND STATE, ENERGY BOUNDS AND WAVE OPERATOR

In this section, we recall some properties that are related to our problem. In [10] the first author proved the following Gagliardo-Nirenberg inequality

$$\||x|^{-b}|u|^4\|_{L_x^1} \leq C_{GN} \|\nabla u\|_{L_x^2}^{3+b} \|u\|_{L_x^2}^{1-b}, \quad (5.1)$$

with the sharp constant (recalling $s_c = \frac{1+b}{2}$)

$$C_{GN} = \frac{4}{3+b} \left(\frac{1-b}{3+b} \right)^{s_c} \frac{1}{\|Q\|_{L^2}^2} \quad (5.2)$$

where Q is the ground state solution of (1.12). Moreover, Q satisfies the following relations

$$\|\nabla Q\|_{L^2}^2 = \frac{3+b}{1-b} \|Q\|_{L^2}^2 \quad (5.3)$$

and

$$\||x|^{-b}|Q|^4\|_{L^1} = \frac{4}{3+b} \|\nabla Q\|_{L^2}^2. \quad (5.4)$$

Note that, combining (5.2), (5.3) and (5.4) one has

$$C_{GN} = \frac{4}{(3+b) \|\nabla Q\|_{L^2}^{2s_c} \|Q\|_{L^2}^{2(1-s_c)}}. \quad (5.5)$$

On the other hand, we also have

$$E[Q] = \frac{1}{2} \|\nabla Q\|_{L^2}^2 - \frac{1}{4} \| |x|^{-b} |Q|^4 \|_{L^1} = \frac{s_c}{3+b} \|\nabla Q\|_{L^2}^2. \quad (5.6)$$

The next lemma provides some estimates that will be needed for the compactness and rigidity results.

Lemma 5.1. *Let $v \in H^1(\mathbb{R}^3)$ such that*

$$\|\nabla v\|_{L^2}^{s_c} \|v\|_{L^2}^{1-s_c} \leq \|\nabla Q\|_{L^2}^{s_c} \|Q\|_{L^2}^{1-s_c}. \quad (5.7)$$

Then, the following statements hold

- (i) $\frac{s_c}{3+b} \|\nabla v\|_{L^2}^2 \leq E(v) \leq \frac{1}{2} \|\nabla v\|_{L^2}^2,$
- (ii) $\|\nabla v\|_{L^2}^{s_c} \|v\|_{L^2}^{1-s_c} \leq w^{\frac{1}{2}} \|\nabla Q\|_{L^2}^{s_c} \|Q\|_{L^2}^{1-s_c},$
- (iii) $16AE[v] \leq 8A \|\nabla v\|_{L^2}^2 \leq 8 \|\nabla v\|_{L^2}^2 - 2(3+b) \| |x|^{-b} |v|^4 \|_{L^1},$

where $w = \frac{E[v]^{s_c} M[v]^{1-s_c}}{E[Q]^{s_c} M[Q]^{1-s_c}}$ and $A = (1-w)$.

Proof. (i) The second inequality is immediate from the definition of Energy (1.3). The first one is obtained by observing that

$$\begin{aligned} E[v] &\geq \frac{1}{2} \|\nabla v\|_{L^2}^2 - \frac{C_{GN}}{4} \|\nabla v\|_{L^2}^{3+b} \|v\|_{L^2}^{1-b} \\ &= \frac{1}{2} \|\nabla v\|_{L^2}^2 \left(1 - \frac{C_{GN}}{2} \|\nabla v\|_{L^2}^{2s_c} \|v\|_{L^2}^{2(1-s_c)} \right) \\ &\geq \frac{1}{2} \|\nabla v\|_{L^2}^2 \left(1 - \frac{C_{GN}}{2} \|\nabla Q\|_{L^2}^{2s_c} \|Q\|_{L^2}^{2(1-s_c)} \right) \\ &= \frac{1+b}{2(3+b)} \|\nabla v\|_{L^2}^2 = \frac{s_c}{3+b} \|\nabla v\|_{L^2}^2, \end{aligned}$$

where we have used (5.1), (5.5) and (5.7).

(ii) The first inequality in (i) yields $\|\nabla v\|_{L^2}^2 \leq \frac{3+b}{s_c} E(v)$, multiplying it by $M[v]^\sigma = \|v\|_{L^2}^{2\sigma}$, where $\sigma = \frac{1-s_c}{s_c}$, we have

$$\begin{aligned} \|\nabla v\|_{L^2}^2 \|v\|_{L^2}^{2\sigma} &\leq \frac{3+b}{s_c} E[v] M[v]^\sigma \\ &= \frac{3+b}{s_c} \frac{E[v] M[v]^\sigma}{E[Q] M[Q]^\sigma} E[Q] M[Q]^\sigma \\ &= w \|\nabla Q\|_{L^2}^2 \|Q\|_{L^2}^{2\sigma}, \end{aligned}$$

where we have used (5.6).

(iii) The first inequality obviously holds. Next, let $B = 8 \|\nabla v\|_{L^2}^2 - 2(3+b) \| |x|^{-b} |v|^4 \|_{L^1}$. Applying the Gagliardo-Nirenberg inequality (5.1) and item (ii) we deduce

$$\begin{aligned} B &\geq 8 \|\nabla v\|_{L^2}^2 - 2(3+b) C_{GN} \|\nabla v\|_{L^2}^{3+b} \|v\|_{L^2}^{1-b} \\ &\geq \|\nabla v\|_{L^2}^2 \left(8 - 2(3+b) C_{GN} w \|\nabla Q\|_{L^2}^{2s_c} \|Q\|_{L^2}^{2(1-s_c)} \right) \\ &= \|\nabla v\|_{L^2}^2 8(1-w), \end{aligned}$$

where in the last equality, we have used (5.5). □

Now, applying the ideas introduced by Côte [4] for the KdV equation (see also Guevara [14] Proposition 2.18, with $(N, \alpha) = (3, 2)$), we show the existence of the Wave Operator. Before stating our result, we prove the following lemma.

Lemma 5.2. *Let $0 < b < 1$. If f and $g \in H^1(\mathbb{R}^3)$ then*

- (i) $\| |x|^{-b} |f|^3 g \|_{L^1} \leq c \|f\|_{L^4}^3 \|g\|_{L^4} + c \|f\|_{L^r}^3 \|g\|_{L^r}$
- (ii) $\| |x|^{-b} |f|^3 g \|_{L^1} \leq c \|f\|_{H^1}^3 \|g\|_{H^1}$

$$(iii) \lim_{|t| \rightarrow +\infty} \| |x|^{-b} |U(t)f|^3 g \|_{L_x^1} = 0.$$

where $\frac{12}{3-b} < r < 6$.

Proof. (i) We divide the estimate in B^C and B . Applying the Hölder inequality, since $1 = \frac{3}{4} + \frac{1}{4}$, one has

$$\begin{aligned} \| |x|^{-b} |f|^3 g \|_{L^1} &\leq \| |x|^{-b} |f|^3 g \|_{L^1(B^C)} + \| |x|^{-b} |f|^3 g \|_{L^1(B)} \\ &\leq \| f \|_{L^4}^3 \| g \|_{L^4} + \| |x|^{-b} \|_{L^\gamma(B)} \| f \|_{L^{3\beta}}^3 \| g \|_{L^r} \\ &= \| f \|_{L^4}^3 \| g \|_{L^4} + \| |x|^{-b} \|_{L^\gamma(B)} \| f \|_{L^r}^3 \| g \|_{L^r}, \end{aligned} \quad (5.8)$$

where

$$1 = \frac{1}{\gamma} + \frac{1}{\beta} + \frac{1}{r} \quad \text{and} \quad r = 3\beta. \quad (5.9)$$

To complete the proof we need to check that $\| |x|^{-b} \|_{L^\gamma(B)}$ is bounded, i.e., $\frac{3}{\gamma} > b$ (see Remark 2.7). In fact, we deduce from (5.9)

$$\frac{3}{\gamma} = 3 - \frac{12}{r},$$

and thus, since $r > \frac{12}{3-b}$ we obtain the desired result ($\frac{3}{\gamma} - b > 0$).

(ii) By the Sobolev inequality (2.4), it is easy to see that $H^1 \hookrightarrow L^4$ and $H^1 \hookrightarrow L^r$ (where $2 < \frac{12}{3-b} < r < 6$), then using (5.8) we get (ii).

(iii) Similarly as (i) and (ii), we get

$$\| |x|^{-b} |U(t)f|^3 g \|_{L_x^1} \leq c \| U(t)f \|_{L^4}^{\alpha+1} \| g \|_{H^1} + c \| U(t)f \|_{L^r}^3 \| g \|_{H^1}, \quad (5.10)$$

for $\frac{12}{3-b} < r < 6$. We now show that $\| U(t)f \|_{L_x^r}$ and $\| U(t)f \|_{L_x^4} \rightarrow 0$ as $|t| \rightarrow +\infty$. Indeed, since r and 4 belong to (2, 6) then it suffices to show

$$\lim_{|t| \rightarrow +\infty} \| U(t)f \|_{L_x^p} = 0, \quad (5.11)$$

where $2 < p < 6$. Let $\tilde{f} \in H^1 \cap L^{p'}$, the Sobolev embedding (2.4) and Lemma 2.1 yield

$$\| U(t)f \|_{L_x^p} \leq c \| f - \tilde{f} \|_{H^1} + c |t|^{-\frac{3(p-2)}{2p}} \| \tilde{f} \|_{L^{p'}}.$$

Since $p > 2$ then the exponent of $|t|$ is negative and so approximating f by $\tilde{f} \in C_0^\infty$ in H^1 , we deduce (5.11). \square

Proposition 5.3. (Existence of Wave Operator) Suppose $\phi \in H^1(\mathbb{R}^3)$ and, for some⁷ $0 < \lambda \leq (\frac{2s_c}{3+b})^{\frac{s_c}{2}}$,

$$\| \nabla \phi \|_{L^2}^{2s_c} \| \phi \|_{L^2}^{2(1-s_c)} < \lambda^2 \left(\frac{3+b}{s_c} \right)^{s_c} E[Q]^{s_c} M[Q]^{1-s_c}. \quad (5.12)$$

Then, there exists $u_0^+ \in H^1(\mathbb{R}^3)$ such that u solving (1.1) with initial data u_0^+ is global in $H^1(\mathbb{R}^3)$ with

- (i) $M[u] = M[\phi]$,
- (ii) $E[u] = \frac{1}{2} \| \nabla \phi \|_{L^2}^2$,
- (iii) $\lim_{t \rightarrow +\infty} \| u(t) - U(t)\phi \|_{H_x^1} = 0$,
- (iv) $\| \nabla u(t) \|_{L^2}^{s_c} \| u(t) \|_{L^2}^{1-s_c} \leq \lambda \| \nabla Q \|_{L^2}^{s_c} \| Q \|_{L^2}^{1-s_c}$.

Proof. We will divide the proof in two parts. First, we construct the wave operator for large time. Indeed, let $I_T = [T, +\infty)$ for $T \gg 1$ and define

$$G(w)(t) = -i \int_t^{+\infty} U(t-s) (|x|^{-b} |w + U(t)\phi|^2 (w + U(t)\phi)(s) ds, \quad t \in I_T$$

and

$$B(T, \rho) = \{ w \in C(I_T; H^1(\mathbb{R}^3)) : \| w \|_T \leq \rho \},$$

where

$$\| w \|_T = \| w \|_{S(\dot{H}^{s_c}; I_T)} + \| w \|_{S(L^2; I_T)} + \| \nabla w \|_{S(L^2; I_T)}.$$

⁷Note that $(\frac{2s_c}{3+b})^{\frac{s_c}{2}} < 1$.

Our goal is to find a fixed point for G on $B(T, \rho)$.

Applying the Strichartz estimates (2.7) (2.8) and Lemmas 4.1-4.3, we deduce

$$\|G(w)\|_{S(\dot{H}^{s_c}; I_T)} \lesssim \|w + U(t)\phi\|_{L_T^\infty H_x^1}^\theta \|w + U(t)\phi\|_{S(\dot{H}^{s_c}; I_T)}^{2-\theta} \|w + U(t)\phi\|_{S(\dot{H}^{s_c}; I_T)} \quad (5.13)$$

$$\|G(w)\|_{S(L^2; I_T)} \lesssim \|w + U(t)\phi\|_{L_T^\infty H_x^1}^\theta \|w + U(t)\phi\|_{S(\dot{H}^{s_c}; I_T)}^{2-\theta} \|w + U(t)\phi\|_{S(L^2; I_T)} \quad (5.14)$$

and

$$\|\nabla G(w)\|_{S(L^2; I_T)} \lesssim \|w + U(t)\phi\|_{L_T^\infty H_x^1}^\theta \|w + U(t)\phi\|_{S(\dot{H}^{s_c}; I_T)}^{2-\theta} \|\nabla(w + U(t)\phi)\|_{S(L^2; I_T)} \quad (5.15)$$

Thus,

$$\|G(w)\|_T \lesssim \|w + U(t)\phi\|_{L_T^\infty H_x^1}^\theta \|w + U(t)\phi\|_{S(\dot{H}^{s_c}; I_T)}^{2-\theta} \|w + U(t)\phi\|_T.$$

Since⁸

$$\|U(t)\phi\|_{S(\dot{H}^{s_c}; I_T)} \rightarrow 0 \quad (5.16)$$

as $T \rightarrow +\infty$, we can find $T_0 > 0$ large enough and $\rho > 0$ small enough such that G is well defined on $B(T_0, \rho)$. The same computations show that G is a contraction on $B(T_0, \rho)$. Therefore, G has a unique fixed point, which we denote by w .

On the other hand, from (5.13) and since

$$\|w + U(t)\phi\|_{L_T^\infty H_x^1} \leq \|w\|_{H^1} + \|\phi\|_{H^1} < +\infty,$$

one has (recalling $G(w) = w$)

$$\begin{aligned} \|w\|_{S(\dot{H}^{s_c}; I_T)} &\lesssim \|w + U(t)\phi\|_{S(\dot{H}^{s_c}; I_T)}^{2-\theta} \|w + U(t)\phi\|_{S(\dot{H}^{s_c}; I_T)} \\ &\lesssim A \|w\|_{S(\dot{H}^{s_c}; I_T)} + A \|U(t)\phi\|_{S(\dot{H}^{s_c}; I_T)} \end{aligned}$$

where $A = \|w + U(t)\phi\|_{S(\dot{H}^{s_c}; I_T)}^{2-\theta}$. In addition, if ρ has been chosen small enough and since $\|U(t)\phi\|_{S(\dot{H}^{s_c}; I_T)}$ is also sufficiently small for T large, we deduce

$$A \leq c \|w\|_{S(\dot{H}^{s_c}; I_T)}^{2-\theta} + c \|U(t)\phi\|_{S(\dot{H}^{s_c}; I_T)}^{2-\theta} < \frac{1}{2},$$

and so (using the last two inequalities)

$$\frac{1}{2} \|w\|_{S(\dot{H}^{s_c}; I_T)} \lesssim A \|U(t)\phi\|_{S(\dot{H}^{s_c}; I_T)},$$

which implies,

$$\|w\|_{S(\dot{H}^{s_c}; I_T)} \rightarrow 0 \quad \text{as } T \rightarrow +\infty. \quad (5.17)$$

Hence, (5.14), (5.15) and (5.17) also yield that⁹

$$\|w\|_{S(L^2; I_T)}, \|\nabla w\|_{S(L^2; I_T)} \rightarrow 0 \quad \text{as } T \rightarrow +\infty,$$

and finally

$$\|w\|_T \rightarrow 0 \quad \text{as } T \rightarrow +\infty. \quad (5.18)$$

Next, we claim that $u(t) = U(t)\phi + w(t)$ satisfies (1.1) in the time interval $[T_0, \infty)$. To do this, we need to show that

$$u(t) = U(t - T_0)u(T_0) + i \int_{T_0}^t U(t - s)(|x|^{-b}|u|^2 u)(s) ds, \quad (5.19)$$

⁸Note that (5.16) is possible not true using the norm $L_{I_T}^\infty L_x^{\frac{6}{3-2s_c}}$ and for this reason we remove the pair $(\infty, \frac{6}{3-2s_c})$ in the definition of \dot{H}^s -admissible pair.

⁹Observe that $\|w + U(t)\phi\|_{S(\dot{H}^{s_c}; I_T)} \leq \|w\|_{S(\dot{H}^{s_c}; I_T)} + \|U(t)\phi\|_{S(\dot{H}^{s_c}; I_T)} \rightarrow 0$ as $T \rightarrow +\infty$ by (5.17) and $\|w + U(t)\phi\|_{L_T^\infty H_x^1}, \|w + U(t)\phi\|_{S(L^2; I_T)}, \|\nabla(w + U(t)\phi)\|_{S(L^2; I_T)} < \infty$ since $w \in B(T, \rho)$ and $\phi \in H^1(\mathbb{R}^3)$.

for all $t \in [T_0, \infty)$. Indeed, since

$$w(t) = -i \int_t^\infty U(t-s)|x|^{-b}|w + U(t)\phi|^2(w + U(t)\phi)(s)ds,$$

then

$$\begin{aligned} U(T_0 - t)w(t) &= -i \int_t^\infty U(T_0 - s)|x|^{-b}|w + U(t)\phi|^2(w + U(t)\phi)(s)ds \\ &= i \int_{T_0}^t U(T_0 - s)|x|^{-b}|w + U(t)\phi|^2(w + U(t)\phi)(s)ds + w(T_0), \end{aligned}$$

and so applying $U(t - T_0)$ on both sides, we get

$$w(t) = U(t - T_0)w(T_0) + i \int_{T_0}^t U(t - s)|x|^{-b}|w + U(t)\phi|^2(w + U(t)\phi)(s)ds.$$

Finally, adding $U(t)\phi$ in both sides of the last equation, we deduce (5.19).

Now we show relations (i)-(iv). Since $u(t) = U(t)\phi + w$ then

$$\|u(t) - U(t)\phi\|_{L_T^\infty H_x^1} = \|w\|_{L_T^\infty H_x^1} \leq c\|w\|_{S(L^2; I_T)} + c\|\nabla w\|_{S(L^2; I_T)} \leq c\|w\|_T \quad (5.20)$$

and so from (5.14) we obtain (iii). Furthermore, using (5.20) it is clear that

$$\lim_{t \rightarrow \infty} \|u(t)\|_{L_x^2} = \|\phi\|_{L^2}. \quad (5.21)$$

and

$$\lim_{t \rightarrow \infty} \|\nabla u(t)\|_{L_x^2} = \|\nabla \phi\|_{L^2}. \quad (5.22)$$

By the mass conservation (1.2) we have $\|u(t)\|_{L^2} = \|u(T_0)\|_{L^2}$ for all t , so from (5.21) we deduce $\|u(T_0)\|_{L^2} = \|\phi\|_{L^2}$, i.e., item (i) holds. On the other hand, it follows from Lemma 5.2 (ii)

$$\begin{aligned} \| |x|^{-b}|u(t)|^4 \|_{L_x^1} &\leq c \| |x|^{-b}|u(t) - U(t)\phi|^4 \|_{L_x^1} + c \| |x|^{-b}|U(t)\phi|^4 \|_{L_x^1} \\ &\leq c \|u(t) - U(t)\phi\|_{H_x^1}^4 + c \| |x|^{-b}|U(t)\phi|^4 \|_{L_x^1}, \end{aligned}$$

which goes to zero as $t \rightarrow +\infty$, by item (iii) and Lemma 5.2 (iii), i.e.

$$\lim_{t \rightarrow \infty} \| |x|^{-b}|u(t)|^4 \|_{L_x^1} = 0. \quad (5.23)$$

Combining (5.22) and (5.23), it is easy to deduce (ii).

Next, in view of (5.12), (i) and (ii) we have

$$E[u]^{s_c} M[u]^{1-s_c} = \frac{1}{2^{s_c}} \|\nabla \phi\|_{L^2}^{2s_c} \|\phi\|_{L^2}^{2(1-s_c)} < \lambda^2 \left(\frac{3+b}{2s_c} \right)^{s_c} E[Q]^{s_c} M[Q]^{1-s_c}$$

and by our choice of λ we conclude

$$E[u]^{s_c} M[u]^{1-s_c} < E[Q]^{s_c} M[Q]^{1-s_c}.$$

Moreover, from (5.21), (5.22) and (5.12)

$$\begin{aligned} \lim_{t \rightarrow \infty} \|\nabla u(t)\|_{L_x^2}^{2s_c} \|u(t)\|_{L_x^2}^{2(1-s_c)} &= \|\nabla \phi\|_{L^2}^{2s_c} \|\phi\|_{L^2}^{2(1-s_c)} \\ &< \lambda^2 \left(\frac{3+b}{s_c} \right)^{s_c} E[Q]^{s_c} M[Q]^{1-s_c} \\ &= \lambda^2 \|\nabla Q\|_{L^2}^{2s_c} \|Q\|_{L^2}^{2(1-s_c)}, \end{aligned}$$

where we have used (5.6). Thus, one can take $T_1 > 0$ sufficiently large such that

$$\|\nabla u(T_1)\|_{L_x^2}^{s_c} \|u(T_1)\|_{L_x^2}^{1-s_c} < \lambda \|\nabla Q\|_{L^2}^{s_c} \|Q\|_{L^2}^{1-s_c}.$$

Therefore, since $\lambda < 1$, we deduce that relations (1.9) and (1.10) hold with $u_0 = u(T_1)$ and so, by Theorem 1.2, we have in fact that $u(t)$ constructed above is a global solution of (1.1). \square

Remark 5.4. *A similar Wave Operator construction also holds when the time limit is taken as $t \rightarrow -\infty$ (backward in time).*

6. EXISTENCE AND COMPACTNESS OF A CRITICAL SOLUTION

The goal of this section is to construct a critical solution (denoted by u_c) of (1.1). We divide the study in two parts, first we establish a profile decomposition result and also an Energy Pythagorean expansion for such decomposition. In the sequel, using the results of the first part we construct u_c and discuss some of its properties.

We start this section recalling some elementary inequalities (see Gérard [13] inequality (1.10) and Guevara [14] page 217). Let $(z_j) \subset \mathbb{C}^M$ with $M \geq 2$. For all $q > 1$ there exists $C_{q,M} > 0$ such that

$$\left| \left| \sum_{j=1}^M z_j \right|^q - \sum_{j=1}^M |z_j|^q \right| \leq C_{q,M} \sum_{j \neq k}^M |z_j| |z_k|^{q-1}, \quad (6.1)$$

and for $\beta > 0$ there exists a constant $C_{\beta,M} > 0$ such that

$$\left| \left| \sum_{j=1}^M z_j \right|^\beta - \sum_{j=1}^M |z_j|^\beta \right| \leq C_{\beta,M} \sum_{j=1}^M \sum_{1 \leq j \neq k \leq M} |z_j|^\beta |z_k|. \quad (6.2)$$

6.1. Profile expansion. This subsection contains a profile decomposition and an energy Pythagorean expansion results. We use similar arguments as the ones in Holmer-Roudenko [17, Lemma 5.2] (see also Fang-Xie-Cazenave [9, Theorem 5.1], with $(N, \alpha) = (3, 2)$) and, for the sake of completeness, we provide the details here.

Proposition 6.1. (Profile decomposition) *Let $\phi_n(x)$ be a radial uniformly bounded sequence in $H^1(\mathbb{R}^3)$. Then for each $M \in \mathbb{N}$ there exists a subsequence of ϕ_n (also denoted by ϕ_n), such that, for each $1 \leq j \leq M$, there exist a profile ψ^j in $H^1(\mathbb{R}^3)$, a sequence t_n^j of time shifts and a sequence W_n^M of remainders in $H^1(\mathbb{R}^3)$, such that*

$$\phi_n(x) = \sum_{j=1}^M U(-t_n^j) \psi^j(x) + W_n^M(x) \quad (6.3)$$

with the properties:

- Pairwise divergence for the time sequences. For $1 \leq k \neq j \leq M$,

$$\lim_{n \rightarrow +\infty} |t_n^j - t_n^k| = +\infty. \quad (6.4)$$

- Asymptotic smallness for the remainder sequence¹⁰

$$\lim_{M \rightarrow +\infty} \left(\lim_{n \rightarrow +\infty} \|U(t) W_n^M\|_{S(\dot{H}^{s_c})} \right) = 0. \quad (6.5)$$

- Asymptotic Pythagorean expansion. For fixed $M \in \mathbb{N}$ and any $s \in [0, 1]$, we have

$$\|\phi_n\|_{\dot{H}^s}^2 = \sum_{j=1}^M \|\psi^j\|_{\dot{H}^s}^2 + \|W_n^M\|_{\dot{H}^s}^2 + o_n(1) \quad (6.6)$$

where $o_n(1) \rightarrow 0$ as $n \rightarrow +\infty$.

¹⁰Recalling that $s_c = \frac{1+b}{2}$.

Proof. Let $C_1 > 0$ such that $\|\phi_n\|_{H^1} \leq C_1$. For every (a, r) \dot{H}^{s_c} -admissible we can define $r_1 = 2r$ and $a_1 = \frac{4r}{r(3-2s_c)-3}$. Note that (a_1, r_1) is also \dot{H}^{s_c} -admissible, then combining the interpolation inequality with $\eta = \frac{3}{r(3-2s_c)-3} \in (0, 1)$ and the Strichartz estimate (2.6), we have

$$\begin{aligned} \|U(t)W_n^M\|_{L_t^a L_x^r} &\leq \|U(t)W_n^M\|_{L_t^{a_1} L_x^{r_1}}^{1-\eta} \|U(t)W_n^M\|_{L_t^\infty L_x^{\frac{6}{3-2s_c}}}^\eta \\ &\leq \|W_n^M\|_{\dot{H}^{s_c}}^{1-\eta} \|U(t)W_n^M\|_{L_t^\infty L_x^{\frac{6}{3-2s_c}}}^\eta. \end{aligned} \quad (6.7)$$

Since we will have $\|W_n^M\|_{\dot{H}^{s_c}} \leq C_1$, then we need to show that the second norm in the right hand side of (6.7) goes to zero as n and M go to infinite, that is

$$\lim_{M \rightarrow +\infty} \left(\limsup_{n \rightarrow +\infty} \|U(t)W_n^M\|_{L_t^\infty L_x^{\frac{6}{3-2s_c}}} \right) = 0. \quad (6.8)$$

First we construct ψ_n^1 , t_n^1 and W_n^1 . Let

$$A_1 = \limsup_{n \rightarrow +\infty} \|U(t)\phi_n\|_{L_t^\infty L_x^{\frac{6}{3-2s_c}}}.$$

If $A_1 = 0$, the proof is complete with $\psi^j = 0$ for all $j = 1, \dots, M$. Assume that $A_1 > 0$. Passing to a subsequence, we may consider $A_1 = \lim_{n \rightarrow +\infty} \|U(t)\phi_n\|_{L_t^\infty L_x^{\frac{6}{3-2s_c}}}$. We claim that there exist a time sequence t_n^1 and ψ^1 such that $U(t_n^1)\phi_n \rightharpoonup \psi^1$ and

$$\beta C_1^{\frac{3-2s_c}{2s_c(1-s_c)}} \|\psi^1\|_{\dot{H}^{s_c}} \geq A_1^{\frac{3-2s_c^2}{2s_c(1-s_c)}}, \quad (6.9)$$

where $\beta > 0$ is independent of C_1 , A_1 and ϕ_n . Indeed, let $\zeta \in C_0^\infty(\mathbb{R}^3)$ a real-valued and radially symmetric function such that $0 \leq \zeta \leq 1$, $\zeta(\xi) = 1$ for $|\xi| \leq 1$ and $\zeta(\xi) = 0$ for $|\xi| \geq 2$. Given $r > 0$, define χ_r by $\widehat{\chi_r}(\xi) = \zeta(\frac{\xi}{r})$. From the Sobolev embedding (2.3) and since the operator $U(t)$ is an isometry in H^{s_c} , we deduce (recalling $0 < s_c < 1$)

$$\begin{aligned} \|U(t)\phi_n - \chi_r * U(t)\phi_n\|_{L_t^\infty L_x^{\frac{6}{3-2s_c}}}^2 &\leq c \|U(t)\phi_n - \chi_r * U(t)\phi_n\|_{L_t^\infty H_x^{s_c}}^2 \\ &\leq c \int |\xi|^{2s_c} |(1 - \widehat{\chi_r})^2| \widehat{\phi_n}(\xi)|^2 d\xi \\ &\leq c \int_{|\xi| > r} |\xi|^{-2(1-s_c)} |\xi|^2 |\widehat{\phi_n}(\xi)|^2 d\xi \\ &\leq cr^{-2(1-s_c)} \|\phi\|_{\dot{H}^1}^2 \leq cr^{-2(1-s_c)} C_1^2. \end{aligned}$$

Choosing

$$r = \left(\frac{4\sqrt{c}C_1}{A_1} \right)^{\frac{1}{1-s_c}} \quad (6.10)$$

and for n large enough we have

$$\|\chi_r * U(t)\phi_n\|_{L_t^\infty L_x^{\frac{6}{3-2s_c}}} \geq \frac{A_1}{2}. \quad (6.11)$$

Note that, from the standard interpolation in Lebesgue spaces

$$\begin{aligned} \|\chi_r * U(t)\phi_n\|_{L_t^\infty L_x^{\frac{6}{3-2s_c}}}^3 &\leq \|\chi_r * U(t)\phi_n\|_{L_t^\infty L_x^2}^{3-2s_c} \|\chi_r * U(t)\phi_n\|_{L_t^\infty L_x^{2s_c}}^{2s_c} \\ &\leq C_1^{3-2s_c} \|\chi_r * U(t)\phi_n\|_{L_t^\infty L_x^{2s_c}}^{2s_c}, \end{aligned} \quad (6.12)$$

thus inequalities (6.11) and (6.12) lead to

$$\|\chi_r * U(t)\phi_n\|_{L_t^\infty L_x^\infty} \geq \left(\frac{A_1}{2C_1^{\frac{3-2s_c}{3}}} \right)^{\frac{3}{2s_c}}.$$

It follows from the radial Sobolev Gagliardo-Nirenberg inequality (since all ϕ_n are radial functions and so are $\chi_r * U(t)\phi_n$) that

$$\|\chi_r * U(t)\phi_n\|_{L_t^\infty L_x^\infty(|x|\geq R)} \leq \frac{1}{R} \|\chi_r * U(t)\phi_n\|_{L_x^2}^{\frac{1}{2}} \|\nabla(\chi_r * U(t)\phi_n)\|_{L_x^2}^{\frac{1}{2}} \leq \frac{C_1}{R},$$

which implies for $R > 0$ sufficiently large

$$\|\chi_r * U(t)\phi_n\|_{L_t^\infty L_x^\infty(|x|\leq R)} \geq \frac{1}{2} \left(\frac{A_1}{2C_1^{\frac{3-2s_c}{3}}} \right)^{\frac{3}{2s_c}},$$

where we have used the two last inequalities. Now, let t_n^1 and x_n^1 , with $|x_n^1| \leq R$, be sequences such that for each $n \in \mathbb{N}$

$$|\chi_r * U(t_n^1)\phi_n(x_n^1)| \geq \frac{1}{4} \left(\frac{A_1}{2C_1^{\frac{3-2s_c}{3}}} \right)^{\frac{3}{2s_c}}$$

or

$$\frac{1}{4} \left(\frac{A_1}{2C_1^{\frac{3-2s_c}{3}}} \right)^{\frac{3}{2s_c}} \leq \left| \int \chi_r(x_n^1 - y) U(t_n^1)\phi_n(y) dy \right|. \quad (6.13)$$

On the other hand, since $\|U(t_n^1)\phi_n\|_{H^1} = \|\phi_n\|_{H^1} \leq C_1$ then $U(t_n^1)\phi_n$ converges weakly in H^1 , i.e., there exists ψ^1 a radial function such that (up to a subsequence) $U(t_n^1)\phi_n \rightharpoonup \psi^1$ in H^1 and $\|\psi^1\|_{H^1} \leq \limsup_{n \rightarrow +\infty} \|\phi_n\|_{H^1} \leq C_1$.

In addition, $x_n^1 \rightarrow x^1$ (also up to a subsequence) since x_n^1 is bounded. Hence the inequality (6.13), the Plancherel formula and the Cauchy-Schwarz inequality yield

$$\frac{1}{8} \left(\frac{A_1}{2C_1^{\frac{3-2s_c}{3}}} \right)^{\frac{3}{2s_c}} \leq \left| \int \chi_r(x^1 - y) \psi^1(y) dy \right| \leq \|\chi_r\|_{\dot{H}^{-s_c}} \|\psi^1\|_{\dot{H}^{s_c}},$$

which implies $\frac{1}{8} \left(\frac{A_1}{2C_1^{\frac{3-2s_c}{3}}} \right)^{\frac{3}{2s_c}} \leq cr^{\frac{3-2s_c}{2}} \|\psi^1\|_{\dot{H}^{s_c}}$, where we have used

$$\|\chi_r\|_{\dot{H}^{-s_c}} = \left(\int_{0 < |\xi| < 2r} |\xi|^{-2s_c} |\widehat{\chi_r}(\xi)|^2 d\xi \right)^{\frac{1}{2}} \leq c \left(\int_0^{2r} \rho^{-2s_c} \rho^2 d\rho \right)^{\frac{1}{2}} \leq cr^{\frac{3-2s_c}{2}}.$$

Therefore in view of our choice of r (see (6.10)) we obtain (6.9), concluding the claim.

Next, define $W_n^1 = \phi_n - U(-t_n^1)\psi^1$. It is easy to see that, for any $0 \leq s \leq 1$,

- $U(t_n^1)W_n^1 \rightharpoonup 0$ in H^1 (since $U(t_n^1)\phi_n \rightharpoonup \psi^1$),
- $\langle \phi_n, U(-t_n^1)\psi^1 \rangle_{\dot{H}^s} = \langle U(t_n^1)\phi_n, \psi^1 \rangle_{\dot{H}^s} \rightarrow \|\psi^1\|_{\dot{H}^s}^2$,
- $\|W_n^1\|_{\dot{H}^s}^2 = \|\phi_n\|_{\dot{H}^s}^2 - \|\psi^1\|_{\dot{H}^s}^2 + o_n(1)$.

The last item, with $s = 0$ and $s = 1$, implies $\|W_n^1\|_{H^1} \leq C_1$.

Let $A_2 = \limsup_{n \rightarrow +\infty} \|U(t)W_n^1\|_{L_t^\infty L_x^{\frac{6}{3-2s}}}$. If $A_2 = 0$ the result follows taking $\psi^j = 0$ for all $j = 2, \dots, M$. Let $A_2 > 0$, repeating the above argument with ϕ_n replaced by W_n^1 we obtain a sequence t_n^2 and a function ψ^2 such that $U(t_n^2)W_n^1 \rightharpoonup \psi^2$ in H^1 and $\beta C_1^{\frac{3-2s_c}{2s_c(1-s_c)}} \|\psi^2\|_{\dot{H}^{s_c}} \geq A_2^{\frac{3-2s_c^2}{2s_c(1-s_c)}}$.

We now prove that $|t_n^2 - t_n^1| \rightarrow +\infty$. In fact, if we suppose (up to a subsequence) $t_n^2 - t_n^1 \rightarrow t^*$ finite, then

$$U(t_n^2 - t_n^1) (U(t_n^1)\phi_n - \psi^1) = U(t_n^2) (\phi_n - U(-t_n^1)\psi^1) = U(t_n^2)W_n^1 \rightharpoonup \psi^2.$$

On the other hand, since $U(t_n^1)\phi_n \rightharpoonup \psi^1$, the left side of the above expression converges weakly to 0, and thus $\psi^2 = 0$, a contradiction. Define $W_n^2 = W_n^1 - U(-t_n^2)\psi^2$. For any $0 \leq s \leq 1$, since $|t_n^1 - t_n^2| \rightarrow +\infty$, we deduce

$$\begin{aligned} \langle \phi_n, U(-t_n^2)\psi^2 \rangle_{\dot{H}^s} &= \langle U(t_n^2)\phi_n, \psi^2 \rangle_{\dot{H}^s} \\ &= \langle U(t_n^2)(W_n^1 + U(-t_n^1)\psi^1), \psi^2 \rangle_{\dot{H}^s} \\ &= \langle U(t_n^2)W_n^1, \psi^2 \rangle_{\dot{H}^s} + \langle U(t_n^2 - t_n^1)\psi^1, \psi^2 \rangle_{\dot{H}^s} \\ &\rightarrow \|\psi^2\|_{\dot{H}^s}^2. \end{aligned}$$

In addition, the definition of W_n^2 implies that

$$\|W_n^2\|_{\dot{H}^s}^2 = \|W_n^1\|_{\dot{H}^{s_c}}^2 - \|\psi^2\|_{\dot{H}^s}^2 + o_n(1)$$

and $\|W_n^2\|_{H^1} \leq C_1$.

By induction we can construct ψ^M , t_n^M and W_n^M such that $U(t_n^M)W_n^{M-1} \rightharpoonup \psi^M$ in H^1 and

$$\beta C_1^{\frac{3-2s_c}{2s_c(1-s_c)}} \|\psi^M\|_{\dot{H}^{s_c}} \geq A_M^{\frac{3-2s_c^2}{2s_c(1-s_c)}}, \quad (6.14)$$

where $A_M = \lim_{n \rightarrow +\infty} \|U(t)W_n^{M-1}\|_{L_t^\infty L_x^{\frac{6}{3-2s_c}}}$.

Next, we show (6.4). Suppose $1 \leq j < M$, we prove that $|t_n^M - t_n^j| \rightarrow +\infty$ by induction assuming $|t_n^M - t_n^k| \rightarrow +\infty$ for $k = j+1, \dots, M-1$. Indeed, let $t_n^M - t_n^j \rightarrow t_0$ finite (up to a subsequence) then it is easy to see

$$\begin{aligned} U(t_n^M - t_n^j)(U(t_n^j)W_n^{j-1} - \psi^j) - U(t_n^M - t_n^{j+1})\psi^{j+1} - \dots - U(t_n^M - t_n^{M-1})\psi^{M-1} \\ = U(t_n^M)W_n^{M-1} \rightharpoonup \psi^M. \end{aligned}$$

Since the left side converges weakly to 0, we have $\psi^M = 0$, a contradiction.

We now consider

$$W_n^M = \phi_n - U(-t_n^1)\psi^1 - U(-t_n^2)\psi^2 - \dots - U(-t_n^M)\psi^M.$$

Similarly as before, by (6.4) we get for any $0 \leq s \leq 1$

$$\langle \phi_n, U(-t_n^M)\psi^M \rangle_{\dot{H}^s} = \langle U(t_n^M)W_n^{M-1}, \psi^M \rangle_{\dot{H}^s} + o_n(1),$$

and so $\langle \phi_n, U(-t_n^M)\psi^M \rangle_{\dot{H}^s} \rightarrow \|\psi^M\|_{\dot{H}^s}^2$. Thus expanding $\|W_n^M\|_{\dot{H}^s}^2$ we deduce that (6.6) also holds.

Finally, the inequality (6.14) together with the relation (6.6) yield

$$\sum_{M \geq 1} \left(\frac{A_M^{\frac{3-2s_c^2}{s_c(1-s_c)}}}{\beta^2 C_1^{\frac{3-2s_c}{s_c(1-s_c)}}} \right) \leq \lim_{n \rightarrow +\infty} \|\phi_n\|_{\dot{H}^{s_c}}^2 < +\infty,$$

which implies that $A_M \rightarrow 0$ as $M \rightarrow +\infty$ i.e., (6.8) holds¹¹. Therefore, from (6.7) we get (6.5). This completes the proof. \square

Remark 6.2. It follows from the proof of Proposition 6.1 that

$$\lim_{M, n \rightarrow \infty} \|W_n^M\|_{L^p} = 0, \quad (6.15)$$

where $2 < p < 6$. Indeed, first we show

$$\lim_{M \rightarrow +\infty} \left(\lim_{n \rightarrow +\infty} \|U(t)W_n^M\|_{L_t^\infty L_x^p} \right) = 0. \quad (6.16)$$

Note that, $\dot{H}^s \hookrightarrow L^p$ where $s = \frac{3}{2} - \frac{3}{p}$ (see inequality (2.3)). Since $2 < p < 6$ then $0 < s < 1$, thus repeating the argument used for showing (6.8) with $\frac{6}{3-2s_c}$ replaced by p and s_c by s , we obtain (6.16). On the other hand, (6.15) follows directly from (6.16) and the inequality

$$\|W_n^M\|_{L_x^p} \leq \|U(t)W_n^M\|_{L_t^\infty L_x^p},$$

since $W_n^M = U(0)W_n^M$.

¹¹ Note that $3 - 2s_c^2 > 0$ since $s_c = \frac{1+b}{2} < 1$.

Proposition 6.3. (Energy Pythagoream Expansion) *Under the hypothesis of Proposition 6.1 we obtain*

$$E[\phi_n] = \sum_{j=1}^M E[U(-t_n^j)\psi^j] + E[W_n^M] + o_n(1). \quad (6.17)$$

Proof. By definition of $E[u]$ and (6.6) with $s = 1$, we have

$$E[\phi_n] - \sum_{j=1}^M E[U(-t_n^j)\psi^j] - E[W_n^M] = -\frac{A_n}{\alpha + 2} + o_n(1),$$

where

$$A_n = \left\| |x|^{-b} |\phi_n|^4 \right\|_{L^1} - \sum_{j=1}^M \left\| |x|^{-b} |U(-t_n^j)\psi^j|^4 \right\|_{L^1_x} - \left\| |x|^{-b} |W_n^M|^4 \right\|_{L^1}.$$

For a fixed $M \in \mathbb{N}$, if $A_n \rightarrow 0$ as $n \rightarrow +\infty$ then (6.17) holds. To prove this fact, pick $M_1 \geq M$ and rewrite the last expression as

$$\begin{aligned} A_n &= \int \left(|x|^{-b} |\phi_n|^4 - \sum_{j=1}^M |x|^{-b} |U(-t_n^j)\psi^j|^4 - |x|^{-b} |W_n^M|^4 \right) dx \\ &= I_n^1 + I_n^2 + I_n^3, \end{aligned}$$

where

$$\begin{aligned} I_n^1 &= \int |x|^{-b} [|\phi_n|^4 - |\phi_n - W_n^{M_1}|^4] dx, \\ I_n^2 &= \int |x|^{-b} [|W_n^{M_1} - W_n^M|^4 - |W_n^M|^4] dx, \\ I_n^3 &= \int |x|^{-b} \left[|\phi_n - W_n^{M_1}|^4 - \sum_{j=1}^M |U(-t_n^j)\psi^j|^4 - |W_n^{M_1} - W_n^M|^4 \right] dx. \end{aligned}$$

We first estimate I_n^1 . Combining (6.1) and Lemma 5.2 (i)-(ii) we have

$$\begin{aligned} |I_n^1| &\lesssim \int |x|^{-b} (|\phi_n|^3 |W_n^{M_1}| + |\phi_n| |W_n^{M_1}|^3 + |W_n^{M_1}|^4) dx \\ &\lesssim (\|\phi_n\|_{L^r}^3 \|W_n^{M_1}\|_{L^r} + \|\phi_n\|_{L^r} \|W_n^{M_1}\|_{L^r}^3 + \|W_n^{M_1}\|_{L^r}^4) + \\ &\quad (\|\phi_n\|_{L^4}^3 \|W_n^{M_1}\|_{L^4} + \|\phi_n\|_{L^4} \|W_n^{M_1}\|_{L^4}^3 + \|W_n^{M_1}\|_{L^4}^4) \\ &\lesssim \|\phi_n\|_{H^1}^3 \|W_n^{M_1}\|_{L^r} + \|\phi_n\|_{H^1} \|W_n^{M_1}\|_{L^r}^3 + \|W_n^{M_1}\|_{L^r}^3 + \\ &\quad \|\phi_n\|_{H^1}^3 \|W_n^{M_1}\|_{L^4} + \|\phi_n\|_{H^1} \|W_n^{M_1}\|_{L^4}^3 + \|W_n^{M_1}\|_{L^4}^4, \end{aligned}$$

where $\frac{12}{3-b} < r < 6$. In view of inequality (6.15) and since $\{\phi_n\}$ is uniformly bounded in H^1 , we conclude that¹²

$$I_n^1 \rightarrow +\infty \text{ as } n, M_1 \rightarrow +\infty.$$

Also, by similar arguments (replacing ϕ_n by W_n^M) we have

$$I_n^2 \rightarrow +\infty \text{ as } n, M_1 \rightarrow +\infty,$$

where we have used that W_n^M is uniformly bounded by (6.6).

Finally we consider the term I_n^3 . Since,

$$\phi_n - W_n^{M_1} = \sum_{j=1}^{M_1} U(-t_n^j)\psi^j \text{ and } W_n^M - W_n^{M_1} = \sum_{j=M+1}^{M_1} U(-t_n^j)\psi^j,$$

¹²We can apply Remark 6.2 since r and $4 \in (2, 6)$.

we can rewrite I_n^3 as

$$\begin{aligned} I_n^3 &= \int |x|^{-b} \left(\left| \sum_{j=1}^{M_1} U(-t_n^j) \psi^j \right|^4 - \sum_{j=1}^{M_1} |U(-t_n^j) \psi^j|^4 \right) dx \\ &\quad - \int |x|^{-b} \left(\left| \sum_{j=M+1}^{M_1} U(-t_n^j) \psi^j \right|^4 - \sum_{j=M+1}^{M_1} |U(-t_n^j) \psi^j|^4 \right) dx. \end{aligned}$$

To complete the prove we make use of the following claim.

Claim. For a fixed $M_1 \in \mathbb{N}$ and for some $j_0 \in \mathbb{N}$ ($j_0 < M_1$), we get

$$D_n = \left\| |x|^{-b} \left| \sum_{j=j_0}^{M_1} U(-t_n^j) \psi^j \right|^4 \right\|_{L_x^1} - \sum_{j=j_0}^{M_1} \| |x|^{-b} |U(-t_n^j) \psi^j|^4 \|_{L_x^1} \rightarrow 0, \text{ as } n \rightarrow +\infty.$$

Indeed, it is clear that the last limit implies that $I_n^3 \rightarrow 0$ as $n \rightarrow +\infty$ completing the proof of relation (6.17).

To prove the claim note that (6.1) implies

$$D_n \leq \sum_{j \neq k}^{M_1} \int |x|^{-b} |U(-t_n^j) \psi^j| |U(-t_n^k) \psi^k|^3 dx.$$

Thus, from Lemma 5.2 (i) one has

$$E_n^{j,k} \leq c \|U(-t_n^k) \psi^k\|_{L_x^4}^3 \|U(-t_n^j) \psi^j\|_{L_x^4} + c \|U(-t_n^k) \psi^k\|_{L_x^r}^3 \|U(-t_n^j) \psi^j\|_{L_x^r},$$

where $2 < \frac{12}{3-b} < r < 6$ and $E_n^{j,k} = \int |x|^{-b} |U(-t_n^j) \psi^j| |U(-t_n^k) \psi^k|^3 dx$. In view of (6.4) we can consider that t_n^k , t_n^j or both go to infinite as n goes to infinite. If $t_n^j \rightarrow +\infty$ as $n \rightarrow +\infty$, so it follow from the last inequality and since $H^1 \hookrightarrow L^4$ and $H^1 \hookrightarrow L^r$ that

$$\begin{aligned} E_n^{j,k} &\leq c \|\psi^k\|_{H^1}^3 \|U(-t_n^j) \psi^j\|_{L_x^4} + c \|\psi^k\|_{H^1}^3 \|U(-t_n^j) \psi^j\|_{L_x^r} \\ &\leq c \|U(-t_n^j) \psi^j\|_{L_x^4} + c \|U(-t_n^j) \psi^j\|_{L_x^r}, \end{aligned}$$

where in the last inequality we have used that $(\psi^k)_{k \in \mathbb{N}}$ is a uniformly bounded sequence in H^1 . Hence, if $n \rightarrow +\infty$ we have $t_n^j \rightarrow +\infty$ and using (5.11) with $t = t_n^j$ and $f = \psi^j$ we conclude that $E_n^{j,k} \rightarrow 0$ as $n \rightarrow +\infty$. Similarly for the case $t_n^k \rightarrow +\infty$ as $n \rightarrow +\infty$, we have

$$\begin{aligned} E_n^{j,k} &\leq c \|U(-t_n^k) \psi^k\|_{L_x^4}^3 \|\psi^j\|_{H^1} + c \|U(-t_n^k) \psi^k\|_{L_x^r}^3 \|\psi^j\|_{H^1} \\ &\leq c \|U(-t_n^k) \psi^k\|_{L_x^4}^3 + c \|U(-t_n^k) \psi^k\|_{L_x^r}^3, \end{aligned}$$

which implies that $E_n^{j,k} \rightarrow 0$ as $n \rightarrow +\infty$ by (5.11) with $t = t_n^k$ and $f = \psi^k$. Finally, since D_n is a finite sum of terms in the form of $E^{j,k}$ we deduce $D_n \rightarrow 0$ as $n \rightarrow +\infty$. \square

6.2. Critical solution. In this subsection, assuming that $\delta_c < E[u]^{s_c} M[u]^{1-s_c}$ (see (3.2)), we construct a global solution, denoted by u_c , of (1.1) with infinite Strichartz norm $\|\cdot\|_{S(\dot{H}^{s_c})}$ and satisfying

$$E[u_c]^{s_c} M[u_c]^{1-s_c} = \delta_c.$$

Next, we show that the flow associated to this critical solution is precompact in $H^1(\mathbb{R}^3)$.

Proposition 6.4. (Existence of a critical solution) *If $\delta_c < E[Q]^{s_c} M[Q]^{1-s_c}$, then there exists a radial function $u_{c,0} \in H^1(\mathbb{R}^3)$ such that the corresponding solution u_c of the IVP (1.1) is global in $H^1(\mathbb{R}^3)$. Moreover the following properties hold*

- (i) $M[u_c] = 1$,
- (ii) $E[u_c]^{s_c} = \delta_c$,
- (iii) $\|\nabla u_{c,0}\|_{L^2}^{s_c} \|u_{c,0}\|_{L^2}^{1-s_c} < \|\nabla Q\|_{L^2}^{s_c} \|Q\|_{L^2}^{1-s_c}$,
- (iv) $\|u_c\|_{S(\dot{H}^{s_c})} = +\infty$.

Proof. Recall from Subsection 3 that there exists a sequence of solutions u_n to (1.1) with H^1 initial data $u_{n,0}$, with $\|u_n\|_{L^2} = 1$ for all $n \in \mathbb{N}$, such that

$$\|\nabla u_{n,0}\|_{L^2}^{s_c} < \|\nabla Q\|_{L^2}^{s_c} \|Q\|_{L^2}^{1-s_c} \quad (6.18)$$

and

$$E[u_n] \searrow \delta_c^{\frac{1}{s_c}} \text{ as } n \rightarrow +\infty.$$

Moreover

$$\|u_n\|_{S(\dot{H}^{s_c})} = +\infty \quad (6.19)$$

for every $n \in \mathbb{N}$. Note that, in view of the assumption $\delta_c < E[Q]^{s_c} M[Q]^{1-s_c}$, there exists $a \in (0, 1)$ such that

$$E[u_n] \leq aE[Q]M[Q]^\sigma, \quad (6.20)$$

where $\sigma = \frac{1-s_c}{s_c}$. Furthermore, (6.18) implies by Lemma 5.1 (ii) that

$$\|\nabla u_{n,0}\|_{L^2}^2 \leq w^{\frac{1}{s_c}} \|\nabla Q\|_{L^2}^2 \|Q\|_{L^2}^{2\sigma},$$

where $w = \frac{E[u_n]^{s_c} M[u_n]^{1-s_c}}{E[Q]^{s_c} M[Q]^{1-s_c}}$, thus we deduce from (6.20) and $\|u_n\|_{L^2} = 1$ that $w^{\frac{1}{s_c}} \leq a$ which implies

$$\|\nabla u_{n,0}\|_{L^2}^2 \leq a \|\nabla Q\|_{L^2}^2 \|Q\|_{L^2}^{2\sigma}. \quad (6.21)$$

On the other hand, the linear profile decomposition (Proposition 6.1) applied to $u_{n,0}$, which is a uniformly bounded sequence in $H^1(\mathbb{R}^3)$ by (6.21), yields

$$u_{n,0}(x) = \sum_{j=1}^M U(-t_n^j) \psi^j(x) + W_n^M(x), \quad (6.22)$$

where M will be taken large later. It follows from the Pythagorean expansion (6.6), with $s = 0$, that for all $M \in \mathbb{N}$

$$\sum_{j=1}^M \|\psi^j\|_{L^2}^2 + \lim_{n \rightarrow +\infty} \|W_n^M\|_{L^2}^2 \leq \lim_{n \rightarrow +\infty} \|u_{n,0}\|_{L^2}^2 = 1, \quad (6.23)$$

this implies that

$$\sum_{j=1}^M \|\psi^j\|_{L^2}^2 \leq 1. \quad (6.24)$$

In addition, another application of (6.6), with $s = 1$, and (6.21) lead to

$$\sum_{j=1}^M \|\nabla \psi^j\|_{L^2}^2 + \lim_{n \rightarrow +\infty} \|\nabla W_n^M\|_{L^2}^2 \leq \lim_{n \rightarrow +\infty} \|\nabla u_{n,0}\|_{L^2}^2 \leq a \|\nabla Q\|_{L^2}^2 \|Q\|_{L^2}^{2\sigma}, \quad (6.25)$$

and so

$$\|\nabla \psi^j\|_{L^2}^{s_c} \leq a^{\frac{s_c}{2}} \|\nabla Q\|_{L^2}^{s_c} \|Q\|_{L^2}^{1-s_c}, \quad j = 1, \dots, M. \quad (6.26)$$

Let $\{t_n^j\}_{n \in \mathbb{N}}$ be the sequence given by Proposition 6.1. From (6.24), (6.26) and the fact that $U(t)$ is an isometry in $L^2(\mathbb{R}^3)$ and $\dot{H}^1(\mathbb{R}^3)$ we deduce

$$\|U(-t_n^j) \psi^j\|_{L_x^2}^{1-s_c} \|\nabla U(-t_n^j) \psi^j\|_{L_x^2}^{s_c} \leq a^{\frac{s_c}{2}} \|\nabla Q\|_{L^2}^{s_c} \|Q\|_{L^2}^{1-s_c}.$$

Now, Lemma 5.1 (i) yields

$$E[U(-t_n^j) \psi^j] \geq c(b) \|\nabla \psi^j\|_{L^2} \geq 0 \quad (6.27)$$

A complete similar analysis also gives, for all $M \in \mathbb{N}$,

$$\begin{aligned} \lim_{n \rightarrow +\infty} \|W_n^M\|_{L^2}^2 &\leq 1, \\ \lim_{n \rightarrow +\infty} \|\nabla W_n^M\|_{L^2}^{s_c} &\leq a^{\frac{s_c}{2}} \|\nabla Q\|_{L^2}^{s_c} \|Q\|_{L^2}^{1-s_c}, \end{aligned}$$

and for n large enough (depending on M)

$$E[W_n^M] \geq 0. \quad (6.28)$$

The energy Pythagorean expansion (Proposition 6.3) allows us to deduce

$$\sum_{j=1}^M \lim_{n \rightarrow +\infty} E[U(-t_n^j) \psi^j] + \lim_{n \rightarrow +\infty} E[W_n^M] = \lim_{n \rightarrow +\infty} E[u_{n,0}] = \delta_c^{\frac{1}{s_c}},$$

which implies, by (6.27) and (6.28), that

$$\lim_{n \rightarrow \infty} E[U(-t_n^j) \psi^j] \leq \delta_c^{\frac{1}{s_c}}, \text{ for all } j = 1, \dots, M. \quad (6.29)$$

Now, if more than one $\psi^j \neq 0$, we show a contradiction and thus the profile expansion given by (6.22) is reduced to the case that only one profile is nonzero. In fact, if more than one $\psi^j \neq 0$, then by (6.23) we must have $M[\psi^j] < 1$ for each j . Passing to a subsequence, if necessary, we have two cases to consider:

Case 1. If for a given j , $t_n^j \rightarrow t^*$ finite (at most only one such j exists by (6.4)), then the continuity of the linear flow in $H^1(\mathbb{R}^3)$ yields

$$U(-t_n^j) \psi^j \rightarrow U(-t^*) \psi^j \text{ strongly in } H^1. \quad (6.30)$$

Let us denote the solution of (1.1) with initial data ψ by $\text{INLS}(t)\psi$. Set $\tilde{\psi}^j = \text{INLS}(t^*)(U(-t^*)\psi^j)$ so that $\text{INLS}(-t^*)\tilde{\psi}^j = U(-t^*)\psi^j$. Since the set

$$\mathcal{K} := \{u_0 \in H^1(\mathbb{R}^3) : \text{relations (1.9) and (1.10) hold}\}$$

is closed in $H^1(\mathbb{R}^3)$ then $\tilde{\psi}^j \in \mathcal{K}$ and therefore $\text{INLS}(t)\tilde{\psi}^j$ is a global solution by Theorem 1.2. Moreover from (6.4), (6.29) and the fact that $M[\psi^j] < 1$ we have

$$\|\tilde{\psi}^j\|_{L_x^2}^{1-s_c} \|\nabla \tilde{\psi}^j\|_{L_x^2}^{s_c} \leq \|\nabla Q\|_{L^2}^{s_c} \|Q\|_{L^2}^{1-s_c}$$

and

$$E[\tilde{\psi}^j]^{s_c} M[\tilde{\psi}^j]^{1-s_c} < \delta_c.$$

So, the definition of δ_c (see (3.2)) implies

$$\|\text{INLS}(t)\tilde{\psi}^j\|_{S(\dot{H}^{s_c})} < +\infty. \quad (6.31)$$

Finally, from (6.30) it is easy to see

$$\lim_{n \rightarrow +\infty} \|\text{INLS}(-t_n^j)\tilde{\psi}^j - U(-t_n^j)\psi^j\|_{H_x^1} = 0. \quad (6.32)$$

Case 2. If $|t_n^j| \rightarrow +\infty$ then by Lemma 5.2 (iii), $\| |x|^{-b} |U(-t_n^j)\psi^j|^4 \|_{L_x^1} \rightarrow 0$. Thus, by the definition of Energy (1.3) and the fact that $U(t)$ is an isometry in $\dot{H}^1(\mathbb{R}^3)$, we deduce

$$\left(\frac{1}{2} \|\nabla \psi^j\|_{L^2}^2 \right)^{s_c} = \lim_{n \rightarrow \infty} E[U(-t_n^j) \psi^j]^{s_c} \leq \delta_c < E[Q]^{s_c} M[Q]^{1-s_c}, \quad (6.33)$$

where we have used (6.29). Therefore, by the existence of wave operator, Proposition 5.3 with $\lambda = (\frac{2s_c}{3+b})^{\frac{s_c}{2}} < 1$ (see also Remark 5.4), there exists $\tilde{\psi}^j \in H^1(\mathbb{R}^3)$ such that

$$M[\tilde{\psi}^j] = M[\psi^j] \quad \text{and} \quad E[\tilde{\psi}^j] = \frac{1}{2} \|\nabla \psi^j\|_{L^2}^2, \quad (6.34)$$

$$\|\nabla \text{INLS}(t)\tilde{\psi}^j\|_{L_x^2}^{s_c} \|\tilde{\psi}^j\|_{L^2}^{1-s_c} < \|\nabla Q\|_{L^2}^{s_c} \|Q\|_{L^2}^{1-s_c} \quad (6.35)$$

and (6.32) also holds in this case.

Since $M[\psi^j] < 1$ and using (6.33)-(6.34), we get $E[\tilde{\psi}^j]^{s_c} M[\tilde{\psi}^j]^{1-s_c} < \delta_c$. Hence, the definition of δ_c together with (6.35) also lead to (6.31).

To sum up, in either case, we obtain a new profile $\tilde{\psi}^j$ for the given ψ^j such that (6.32) (6.31) hold.

Next, we define $u_n(t) = \text{INLS}(t)u_{n,0}$; $v^j(t) = \text{INLS}(t)\tilde{\psi}^j$; $\tilde{u}_n(t) = \sum_{j=1}^M v^j(t - t_n^j)$ and

$$\tilde{W}_n^M = \sum_{j=1}^M \left[U(-t_n^j) \psi^j - \text{INLS}(-t_n^j) \tilde{\psi}^j \right] + W_n^M. \quad (6.36)$$

Then $\tilde{u}_n(t)$ solves the following equation

$$i\partial_t \tilde{u}_n + \Delta \tilde{u}_n + |x|^{-b} |\tilde{u}_n|^2 \tilde{u}_n = e_n^M, \quad (6.37)$$

where

$$e_n^M = |x|^{-b} \left(|\tilde{u}_n|^2 \tilde{u}_n - \sum_{j=1}^M |v^j(t - t_n^j)|^2 v^j(t - t_n^j) \right). \quad (6.38)$$

Also note that by definition of \tilde{W}_n^M in (6.36) and (6.22) we can write

$$u_{n,0} = \sum_{j=1}^M \text{INLS}(-t_n^j) \tilde{\psi}^j + \tilde{W}_n^M,$$

so it is easy to see $u_{n,0} - \tilde{u}_n(0) = \tilde{W}_n^M$, then combining (6.36) and the Strichartz inequality (2.6), we estimate

$$\|U(t) \tilde{W}_n^M\|_{S(\dot{H}^{s_c})} \leq c \sum_{j=1}^M \|\text{INLS}(-t_n^j) \tilde{\psi}^j - U(-t_n^j) \psi^j\|_{H^1} + \|U(t) W_n^M\|_{S(\dot{H}^{s_c})},$$

which implies

$$\lim_{M \rightarrow +\infty} \left[\lim_{n \rightarrow +\infty} \|U(t)(u_{n,0} - \tilde{u}_{n,0})\|_{S(\dot{H}^{s_c})} \right] = 0, \quad (6.39)$$

where we used (6.5) and (6.32).

The idea now is to approximate u_n by \tilde{u}_n . Therefore, from the long time perturbation theory (Proposition 4.10) and (6.31) we conclude $\|u_n\|_{S(\dot{H}^{s_c})} < +\infty$, for n large enough, which is a contradiction with (6.19). Indeed, we assume the following two claims for a moment to conclude the proof.

Claim 1. For each M and $\varepsilon > 0$, there exists $n_0 = n_0(M, \varepsilon)$ such that

$$n > n_0 \Rightarrow \|e_n^M\|_{S'(\dot{H}^{-s_c})} + \|e_n^M\|_{S'(L^2)} + \|\nabla e_n^M\|_{S'(L^2)} \leq \varepsilon. \quad (6.40)$$

Claim 2. There exist $L > 0$ and $S > 0$ independent of M such that for any M , there exists $n_1 = n_1(M)$ such that

$$n > n_1 \Rightarrow \|\tilde{u}_n\|_{S(\dot{H}^{s_c})} \leq L \text{ and } \|\tilde{u}_n\|_{L_t^\infty H_x^1} \leq S. \quad (6.41)$$

Note that by (6.39), there exists $M_1 = M_1(\varepsilon)$ such that for each $M > M_1$ there exists $n_2 = n_2(M)$ such that

$$n > n_2 \Rightarrow \|U(t)(u_{n,0} - \tilde{u}_{n,0})\|_{S(\dot{H}^{s_c})} \leq \varepsilon,$$

with $\varepsilon < \varepsilon_1$ as in Proposition 4.10. Thus, if the two claims hold true, by Proposition 4.10, for M large enough and $n > \max\{n_0, n_1, n_2\}$, we obtain $\|u_n\|_{S(\dot{H}^{s_c})} < +\infty$, reaching the desired contradiction.

Up to now, we have reduced the profile expansion to the case where $\psi^1 \neq 0$ and $\psi^j = 0$ for all $j \geq 2$. We now begin to show the existence of a critical solution. From the same arguments as the ones in the previous case (the case when more than one $\psi^j \neq 0$), we can find $\tilde{\psi}^1$ such that

$$u_{n,0} = \text{INLS}(-t_n^1) \tilde{\psi}^1 + \tilde{W}_n^M,$$

with

$$M[\tilde{\psi}^1] = M[\psi^1] \leq 1 \quad (6.42)$$

$$E[\tilde{\psi}^1]^{s_c} = \left(\frac{1}{2} \|\nabla \psi^1\|_{L^2}^2 \right)^{s_c} \leq \delta_c \quad (6.43)$$

$$\|\nabla \text{INLS}(t) \tilde{\psi}^1\|_{L_x^{s_c}}^{s_c} \|\tilde{\psi}^1\|_{L^2}^{1-s_c} < \|\nabla Q\|_{L^2}^{s_c} \|Q\|_{L^2}^{1-s_c} \quad (6.44)$$

and

$$\lim_{n \rightarrow +\infty} \|U(t)(u_{n,0} - \tilde{u}_{n,0})\|_{S(\dot{H}^{s_c})} = \lim_{n \rightarrow +\infty} \|U(t) \tilde{W}_n^M\|_{S(\dot{H}^{s_c})} = 0. \quad (6.45)$$

Let $\tilde{\psi}^1 = u_{c,0}$ and u_c be the global solution to (1.1) (in view of Theorem 1.2 and inequalities (6.42)-(6.44)) with initial data $\tilde{\psi}^1$, that is, $u_c(t) = \text{INLS}(t) \tilde{\psi}^1$. We claim that

$$\|u_c\|_{S(\dot{H}^{s_c})} = +\infty. \quad (6.46)$$

Indeed, suppose, by contradiction, that $\|u_c\|_{S(\dot{H}^{s_c})} < +\infty$. Let,

$$\tilde{u}_n(t) = \text{INLS}(t - t_n^j) \tilde{\psi}^1,$$

then

$$\|\tilde{u}_n(t)\|_{S(\dot{H}^{s_c})} = \|\text{INLS}(t - t_n^j) \tilde{\psi}^1\|_{S(\dot{H}^{s_c})} = \|\text{INLS}(t) \tilde{\psi}^1\|_{S(\dot{H}^{s_c})} = \|u_c\|_{S(\dot{H}^{s_c})} < +\infty.$$

Furthermore, it follows from (6.42)-(6.45) that

$$\sup_{t \in \mathbb{R}} \|\tilde{u}_n\|_{H_x^1} = \sup_{t \in \mathbb{R}} \|u_c\|_{H_x^1} < +\infty.$$

and

$$\|U(t)(u_{n,0} - \tilde{u}_{n,0})\|_{S(\dot{H}^{s_c})} \leq \varepsilon,$$

for n large enough. Hence, by the long time perturbation theory (Proposition 4.10) with $e = 0$, we obtain $\|u_n\|_{S(\dot{H}^{s_c})} < +\infty$, which is a contradiction with (6.19).

On the other hand, the relation (6.46) implies $E[u_c]^{s_c} M[u_c]^{1-s_c} = \delta_c$ (see (3.2)). Thus, we conclude from (6.42) and (6.43)

$$M[u_c] = 1 \quad \text{and} \quad E[u_c]^{s_c} = \delta_c.$$

Also note that (6.44) implies (iii) in the statement of the Proposition 6.4.

To complete the proof it remains to establish Claims 1 and 2 (see (6.41) and (6.40)).

Proof of Claim 1. First, we show that for each M and $\varepsilon > 0$, there exists $n_0 = n_0(M, \varepsilon)$ such that $\|e_n^M\|_{S'(\dot{H}^{-s_c})} < \frac{\varepsilon}{3}$. From (6.38) and (6.2) (with $\beta = 2$), we deduce

$$\|e_n^M\|_{S'(\dot{H}^{-s_c})} \leq C_{\alpha, M} \sum_{j=1}^M \sum_{1 \leq j \neq k \leq M} \| |x|^{-b} |v^k|^2 |v^j| \|_{L_t^{\tilde{a}'} L_x^{\hat{r}'}}. \quad (6.47)$$

We claim that the norm in the right hand side of (6.47) goes to 0 as $n \rightarrow +\infty$. Indeed, by the relation (4.8) one has

$$\| |x|^{-b} |v^k|^2 |v^j| \|_{L_t^{\tilde{a}'} L_x^{\hat{r}'}} \leq c \|v^k\|_{L_t^\theta H_x^1}^\theta \|v^k(t - t_n^k)\|_{L_x^{\hat{r}}}^{2-\theta} \|v^j(t - t_n^j)\|_{L_x^{\hat{r}}} \|v^j\|_{L_t^{\tilde{a}'} L_x^{\hat{r}}}. \quad (6.48)$$

Fix $1 \leq j \neq k \leq M$. Note that, $\|v^k\|_{H_x^1} < +\infty$ (see (6.34) - (6.35)) and by (6.31) $v^j, v^k \in S(\dot{H}^{s_c})$ and, so we can approximate v^j by functions of $C_0^\infty(\mathbb{R}^{3+1})$. Hence, defining

$$g_n(t) = \|v^k(t)\|_{L_x^{\hat{r}}}^{2-\theta} \|v^j(t - (t_n^j - t_n^k))\|_{L_x^{\hat{r}}},$$

we deduce

(i) $g_n \in L_t^{\tilde{a}'}$. Indeed, applying the Hölder inequality since $\frac{1}{\tilde{a}'} = \frac{\alpha-\theta}{\tilde{a}} + \frac{1}{\tilde{a}}$ we get

$$\|g_n\|_{L_t^{\tilde{a}'}} \leq \|v^k\|_{L_t^{\tilde{a}} L_x^{\hat{r}}}^{2-\theta} \|v^j\|_{L_t^{\tilde{a}} L_x^{\hat{r}}} \leq \|v^k\|_{S(\dot{H}^{s_c})}^{2-\theta} \|v^j\|_{S(\dot{H}^{s_c})} < +\infty.$$

Furthermore, (6.4) implies that $g_n(t) \rightarrow 0$ as $n \rightarrow +\infty$.

(ii) $|g_n(t)| \leq K I_{\text{supp}(v^j)} \|v^k(t)\|_{L_x^{\hat{r}}}^{2-\theta} \equiv g(t)$ for all n , where $K > 0$ and $I_{\text{supp}(v^j)}$ is the characteristic function of $\text{supp}(v^j)$. Similarly as (i), we obtain

$$\|g\|_{L_t^{\tilde{a}'}} \leq \|v^k\|_{L_t^{\tilde{a}} L_x^{\hat{r}}}^{2-\theta} \|I_{\text{supp}(v^j)}\|_{L_t^{\tilde{a}} L_x^{\hat{r}}} < +\infty.$$

That is, $g \in L_t^{\tilde{a}'}$.

Then, the Dominated Convergence Theorem yields $\|g_n\|_{L_t^{\tilde{a}'}} \rightarrow 0$ as $n \rightarrow +\infty$, and so combining this result with (6.48) we conclude the proof of the first estimate.

Next, we prove $\|e_n^M\|_{S'(L^2)} < \frac{\varepsilon}{3}$. Using again the elementary inequality (6.2) we estimate

$$\|e_n^M\|_{S'(L^2)} \leq C_{\alpha, M} \sum_{j=1}^M \sum_{1 \leq j \neq k \leq M} \| |x|^{-b} |v^k|^2 |v^j| \|_{L_t^{\tilde{a}'} L_x^{\hat{r}'}}.$$

On the other hand, we have (see proof of Lemma 4.1 (ii))

$$\begin{aligned}
\| |x|^{-b} |v^k|^2 |v^j| \|_{L_t^{\hat{q}'} L_x^{\hat{r}'}} &\leq c \|v^k\|_{L_t^\infty H_x^1}^\theta \left\| \|v^k(t - t_n^k)\|_{L_x^{\hat{r}}}^{2-\theta} \|v^j(t - t_n^j)\|_{L_x^{\hat{r}}} \right\|_{L_t^{\hat{q}'}} \\
&\leq c \|v^k\|_{L_t^\infty H_x^1}^\theta \|v^k\|_{L_t^{\hat{q}} L_x^{\hat{r}}}^{2-\theta} \|v^j\|_{L_t^{\hat{q}} L_x^{\hat{r}}} \\
&\leq c \|v^k\|_{L_t^\infty H_x^1}^\theta \|v^k\|_{S(\dot{H}^{s_c})}^{2-\theta} \|v^j\|_{S(L^2)}.
\end{aligned}$$

Since $v^j \in S(\dot{H}^{s_c})$ then by (4.22) the norms $\|v^j\|_{S(L^2)}$ and $\|\nabla v^j\|_{S(L^2)}$ are bounded quantities. This implies that the right hand side of the last inequality is finite. Therefore, using the same argument as in the previous case we get

$$\left\| \|v^k(t - t_n^k)\|_{L_x^{\hat{r}}}^{2-\theta} \|v^j(t - t_n^j)\|_{L_x^{\hat{r}}} \right\|_{L_t^{\hat{q}'}} \rightarrow 0,$$

as $n \rightarrow +\infty$, which lead to $\| |x|^{-b} |v^k|^2 |v^j| \|_{L_t^{\hat{q}'} L_x^{\hat{r}'}} \rightarrow 0$.

Finally, we prove $\|\nabla e_n^M\|_{S'(L^2)} < \frac{\varepsilon}{3}$. Note that

$$\begin{aligned}
\nabla e_n^M &= \nabla(|x|^{-b}) \left(f(\tilde{u}_n) - \sum_{j=1}^M f(v^j) \right) + |x|^{-b} \nabla \left(f(\tilde{u}_n) - \sum_{j=1}^M f(v^j) \right) \\
&\equiv R_n^1 + R_n^2,
\end{aligned} \tag{6.49}$$

where $f(v) = |v|^2 v$. First, we consider R_n^1 . The estimate (6.2) yields

$$\|R_n^1\|_{S'(L^2)} \leq c C_{\alpha, M} \sum_{j=1}^M \sum_{1 \leq j \neq k \leq M} \| |x|^{-b-1} |v^k|^2 |v^j| \|_{L_t^{\hat{q}'} L_x^{\hat{r}'}}$$

and by Remark 4.5 we deduce that $\| |x|^{-b-1} |v^k|^2 |v^j| \|_{L_t^{\hat{q}'} L_x^{\hat{r}'}}$ is finite, then by the same argument as before we have

$$\| |x|^{-b-1} |v^k(t - t_n^k)|^2 |v^j(t - t_n^j)| \|_{L_t^{\hat{q}'} L_x^{\hat{r}'}} \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

Therefore, the last two relations yield $\|R_n^1\|_{S'(L^2)} \rightarrow 0$ as $n \rightarrow +\infty$.

On the other hand, observe that

$$\begin{aligned}
\nabla(f(\tilde{u}_n) - \sum_{j=1}^M f(v^j)) &= f'(\tilde{u}_n) \nabla \tilde{u}_n - \sum_{j=1}^M f'(v^j) \nabla v^j \\
&= \sum_{j=1}^M (f'(\tilde{u}_n) - f'(v^j)) \nabla v^j.
\end{aligned} \tag{6.50}$$

Since $|f'(\tilde{u}_n) - f'(v^j)| \leq C_{\alpha, M} \sum_{1 \leq k \neq j \leq M} |v^k|(|v^j| + |v^k|)$, we deduce using the last two relations together with (6.49) and (6.50)

$$\|R_n^2\|_{S'(L^2)} \lesssim \sum_{j=1}^M \sum_{1 \leq k \neq j \leq M} \| |x|^{-b} |v^k|(|v^j| + |v^k|) |\nabla v^j| \|_{S'(L^2)}.$$

Therefore, from Lemma 4.1 (ii) (see also Remark 4.2) we have that the right hand side of the last two inequalities are finite quantities and, by an analogous argument as before, we conclude that

$$\|R_n^2\|_{S'(L^2)} \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

This completes the proof of Claim 1.

Proof of Claim 2. First, we show that $\|\tilde{u}_n\|_{L_t^\infty H_x^1}$ and $\|\tilde{u}_n\|_{L_t^\gamma L_x^\gamma}$ are bounded quantities where $\gamma = \frac{10}{3}$. Indeed, we already know (see (6.24) and (6.25)) that there exists C_0 such that

$$\sum_{j=1}^{\infty} \|\psi^j\|_{H_x^1}^2 \leq C_0,$$

then we can choose $M_0 \in \mathbb{N}$ large enough such that

$$\sum_{j=M_0}^{\infty} \|\psi^j\|_{H_x^1}^2 \leq \frac{\delta}{2}, \quad (6.51)$$

where $\delta > 0$ is a sufficiently small.

Fix $M \geq M_0$. From (6.32), there exists $n_1(M) \in \mathbb{N}$ where for all $n > n_1(M)$, we obtain

$$\sum_{j=M_0}^M \|\text{INLS}(-t_n^j) \tilde{\psi}^j\|_{H_x^1}^2 \leq \delta,$$

where we have used (6.51). This is equivalent to

$$\sum_{j=M_0}^M \|v^j(-t_n^j)\|_{H_x^1}^2 \leq \delta. \quad (6.52)$$

Therefore, by the Small Data Theory (Proposition 4.6)¹³

$$\sum_{j=M_0}^M \|v^j(t - t_n^j)\|_{L_t^\infty H_x^1}^2 \leq c\delta \text{ for } n \geq n_1(M).$$

Note that,

$$\left\| \sum_{j=M_0}^M v^j(t - t_n^j) \right\|_{H_x^1}^2 = \sum_{j=M_0}^M \|v^j(t - t_n^j)\|_{H_x^1}^2 + 2 \sum_{M_0 \leq l \neq k \leq M} \langle v^l(t - t_n^l), v^k(t - t_n^k) \rangle_{H_x^1},$$

so, for $l \neq k$ we deduce from (6.4) that (see [9, Corollary 4.4] for more details)

$$\sup_{t \in \mathbb{R}} |\langle v^l(t - t_n^l), v^k(t - t_n^k) \rangle_{H_x^1}| \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

Hence, since $\|v^j\|_{L_t^\infty H_x^1}$ is bounded (see (6.34) - (6.35)), by definition of \tilde{u}_n there exists $S > 0$ (independent of M) such that

$$\sup_{t \in \mathbb{R}} \|\tilde{u}_n\|_{H_x^1}^2 \leq S \text{ for } n > n_1(M). \quad (6.53)$$

We now show $\|\tilde{u}_n\|_{L_t^\gamma L_x^\gamma} \leq L_1$. Using again (6.52) with δ small enough and the Small Data Theory (noting that (γ, γ) is L^2 -admissible and $\gamma > 2$), we have

$$\sum_{j=M_0}^M \|v^j(t - t_n^j)\|_{L_t^\gamma L_x^\gamma}^\gamma \leq c \sum_{j=M_0}^M \|v^j(-t_n^j)\|_{H_x^1}^\gamma \leq c \sum_{j=M_0}^M \|v^j(-t_n^j)\|_{H_x^1}^2 \leq c\delta, \quad (6.54)$$

for $n \geq n_1(M)$.

On the other hand, in view of (6.1)

$$\left\| \sum_{j=M_0}^M v^j(t - t_n^j) \right\|_{L_t^\gamma L_x^\gamma}^\gamma \leq \sum_{j=M_0}^M \|v^j\|_{L_t^\gamma L_x^\gamma}^\gamma + C_M \sum_{M_0 \leq j \neq k \leq M} \int_{\mathbb{R}^{3+1}} |v^j| |v^k|^{\gamma-2}$$

for all $M > M_0$. Observe that, given j such that $M_0 \leq j \neq k \leq M$, the Hölder inequality yields

$$\begin{aligned} \int_{\mathbb{R}^{3+1}} |v^j| |v^k|^{\gamma-2} &\leq \|v^k(t - t_n^k)\|_{L_t^\gamma L_x^\gamma} \left(\int_{\mathbb{R}^{3+1}} |v^j|^{\frac{\gamma}{2}} |v^k|^{\frac{\gamma}{2}} \right)^{\frac{2}{\gamma}} \\ &\leq c \|v^j(-t_n^j)\|_{H_x^1} \left(\int_{\mathbb{R}^{3+1}} |v^j|^{\frac{\gamma}{2}} |v^k|^{\frac{\gamma}{2}} \right)^{\frac{2}{\gamma}}. \end{aligned} \quad (6.55)$$

Since v^j and $v^k \in L_t^\gamma L_x^\gamma$ we have that the right hand side of (6.55) is bounded and so by similar arguments as in the previous claim, we deduce from (6.4) that the integral in the right hand side of the previous inequality

¹³Recall that the pair $(\infty, 2)$ is L^2 -admissible.

goes to 0 as $n \rightarrow +\infty$ (another proof of this fact can be found in [9, Lemma 4.5]). This implies that there exists L_1 (independent of M) such that

$$\|\tilde{u}_n\|_{L_t^\gamma L_x^\gamma} \leq \sum_{j=1}^{M_0} \|v^j\|_{L_t^\gamma L_x^\gamma} + \left\| \sum_{j=M_0}^M v^j \right\|_{L_t^\gamma L_x^\gamma} \leq L_1 \quad \text{for } n \geq n_1(M), \quad (6.56)$$

where we have used (6.54).

To complete the proof of the Claim 2 we will show the following inequality

$$\|\tilde{u}_n\|_{L_t^{\hat{a}} L_x^{\hat{r}}} \leq \|\tilde{u}_n\|_{L_t^\infty H_x^1}^{1-\frac{\gamma}{\hat{a}}} \|\tilde{u}_n\|_{L_t^\gamma L_x^\gamma}^{\frac{\gamma}{\hat{a}}}, \quad (6.57)$$

where \hat{a} and \hat{r} are defined in (4.1)-(4.2).

Assuming the last inequality for a moment let us conclude the proof of the Claim 2. Indeed combining (6.53) and (6.56) we deduce from (6.57) that

$$\|\tilde{u}_n\|_{L_t^{\hat{a}} L_x^{\hat{r}}} \leq S^{1-\frac{\gamma}{\hat{a}}} L_1^{\frac{\gamma}{\hat{a}}} = L_2, \quad \text{for } n \geq n_1(M).$$

Then, since \tilde{u}_n satisfies the perturbed equation (6.37) we can apply the Strichartz estimates to the integral formulation and conclude (using also Claim 1)

$$\begin{aligned} \|\tilde{u}_n\|_{S(\dot{H}^{s_c})} &\leq c\|\tilde{u}_{n,0}\|_{H_x^1} + c\||x|^{-b}|\tilde{u}_n|^2\tilde{u}_n\|_{L_t^{\hat{a}'} L_x^{\hat{r}'}} + \|e_n^M\|_{S'(\dot{H}^{-s_c})} \\ &\leq cS + cL_2 + \varepsilon = L, \end{aligned}$$

for $n \geq n_1(M)$, which completes the proof of the Claim 2.

We now prove the inequality (6.57). Indeed, the interpolation inequality implies

$$\|\tilde{u}_n\|_{L_t^{\hat{a}} L_x^{\hat{r}}} \leq \|\tilde{u}_n\|_{L_t^\infty L_x^p}^{1-\frac{\gamma}{\hat{a}}} \|\tilde{u}_n\|_{L_t^\gamma L_x^\gamma}^{\frac{\gamma}{\hat{a}}},$$

where $\frac{1}{\hat{r}} = (1 - \frac{\gamma}{\hat{a}}) \left(\frac{1}{p}\right) + \frac{1}{\hat{a}}$. Using the values of \hat{r} , \hat{a} and γ we obtain

$$p = \frac{14 - 6\theta + 10b}{3 + b - \theta(2 - b)}.$$

Choosing $0 < \theta < 2/3$ and $b < 1$ then it is easy to see that $2 < p < 6$. Thus by the Sobolev embedding $H^1 \hookrightarrow L^p$ the inequality (6.57) holds. \square

In the next proposition, we prove the precompactness of the flow associated to the critical solution u_c . The argument is very similar to Holmer-Roudenko [17, Proposition 5.5].

Proposition 6.5. (Precompactness of the flow of the critical solution) *Let u_c be as in Proposition 6.4 and define*

$$K = \{u_c(t) : t \in [0, +\infty)\} \subset H^1.$$

Then K is precompact in $H^1(\mathbb{R}^3)$.

Proof. Let $\{t_n\} \subseteq [0, +\infty)$ a sequence of times and $\phi_n = u_c(t_n)$ be a uniformly bounded sequence in $H^1(\mathbb{R}^3)$. We need to show that $u_c(t_n)$ has a subsequence converging in $H^1(\mathbb{R}^3)$. If $\{t_n\}$ is bounded, we can assume $t_n \rightarrow t^*$ finite, so by the continuity of the solution in $H^1(\mathbb{R}^3)$ the result is clear. Next, assume that $t_n \rightarrow +\infty$.

The linear profile expansion (Proposition 6.1) implies the existence of profiles ψ^j and a remainder W_n^M such that

$$u_c(t_n) = \sum_{j=1}^M U(-t_n^j) \psi^j + W_n^M,$$

with $|t_n^j - t_n^k| \rightarrow +\infty$ as $n \rightarrow +\infty$ for any $j \neq k$. Then, by the energy Pythagorean expansion (Proposition 6.3), we get

$$\sum_{j=1}^M \lim_{n \rightarrow +\infty} E[U(-t_n^j) \psi^j] + \lim_{n \rightarrow +\infty} E[W_n^M] = E[u_c] = \delta_c, \quad (6.58)$$

where we have used Proposition 6.4 (ii). This implies that

$$\lim_{n \rightarrow +\infty} E[U(-t_n^j)\psi^j] \leq \delta_c \quad \forall j,$$

since each energy in (6.58) is nonnegative by Lemma (5.1) (i).

Moreover, by (6.6) with $s = 0$ we obtain

$$\sum_{j=1}^M M[\psi^j] + \lim_{n \rightarrow +\infty} M[W_n^M] = M[u_c] = 1, \quad (6.59)$$

by Proposition 6.4 (i).

If more than one $\psi^j \neq 0$, similar to the proof in Proposition 6.4, we have a contradiction with the fact that $\|u_c\|_{S(\dot{H}^{s_c})} = +\infty$. Thus, we address the case that only $\psi^j = 0$ for all $j \geq 2$, and so

$$u_c(t_n) = U(-t_n^1)\psi^1 + W_n^M. \quad (6.60)$$

Also as in the proof of Proposition 6.4, we obtain that

$$M[\psi^1] = M[u_c] = 1 \quad \text{and} \quad \lim_{n \rightarrow +\infty} E[U(-t_n^1)\psi^1] = \delta_c, \quad (6.61)$$

and using (6.58), (6.59) together with (6.61), we deduce that

$$\lim_{n \rightarrow +\infty} M[W_n^M] = 0 \quad \text{and} \quad \lim_{n \rightarrow +\infty} E[W_n^M] = 0. \quad (6.62)$$

Thus, Lemma 5.1 (i) yields

$$\lim_{n \rightarrow +\infty} \|W_n^M\|_{H^1} = 0. \quad (6.63)$$

We claim now that t_n^1 converges to some finite t^* (up to a subsequence). In this case, since $U(-t_n^1)\psi^1 \rightarrow U(-t^*)\psi^1$ in $H^1(\mathbb{R}^3)$ and (6.63) holds, the relation (6.60) implies that $u_c(t_n)$ converges in $H^1(\mathbb{R}^3)$, concluding the proof.

Assume by contradiction that $|t_n^1| \rightarrow +\infty$, then we have two cases to consider. If $t_n^1 \rightarrow -\infty$, by (6.60)

$$\|U(t)u_c(t_n)\|_{S(\dot{H}^{s_c};[0,+\infty))} \leq \|U(t-t_n^1)\psi^1\|_{S(\dot{H}^{s_c};[0,+\infty))} + \|U(t)W_n^M\|_{S(\dot{H}^{s_c};[0,+\infty))}.$$

Next, note that since $t_n^1 \rightarrow -\infty$ we obtain

$$\|U(t-t_n^1)\psi^1\|_{S(\dot{H}^{s_c};[0,+\infty))} \leq \|U(t)\psi^1\|_{S(\dot{H}^{s_c};[-t_n^1,+\infty))} \leq \frac{1}{2}\delta,$$

and also

$$\|U(t)W_n^M\|_{S(\dot{H}^{s_c})} \leq \frac{1}{2}\delta,$$

given $\delta > 0$ for n, M sufficiently large, where in the last inequality we have used (2.6) and (6.63). Hence,

$$\|U(t)u_c(t_n)\|_{S(\dot{H}^{s_c};[0,+\infty))} \leq \delta.$$

Therefore, choosing $\delta > 0$ sufficiently small, by the small data theory (Proposition 4.6) we get that $\|u_c\|_{S(\dot{H}^{s_c})} \leq 2\delta$, which is a contradiction with Proposition 6.4 (iv).

On the other hand, if $t_n^1 \rightarrow +\infty$, the same arguments also give that for n large,

$$\|U(t)u_c(t_n)\|_{S(\dot{H}^{s_c};(-\infty,0])} \leq \delta,$$

and again the small data theory (Proposition 4.6) implies $\|u_c\|_{S(\dot{H}^{s_c};(-\infty,t_n])} \leq 2\delta$. Since $t_n \rightarrow +\infty$ as $n \rightarrow +\infty$, from the last inequality we get $\|u_c\|_{S(\dot{H}^{s_c})} \leq 2\delta$, which is also a contradiction. Thus, t_n^1 must converge to some finite t^* , completing the proof of Proposition 6.5.

□

7. RIGIDITY THEOREM

The main result of this section is a rigidity theorem, which implies that the critical solution u_c constructed in Section 6.2 must be identically zero and so reaching a contradiction in view of Proposition 6.4 (iv). Before proving this result, we begin showing some preliminaries that will help us in the proof.

Proposition 7.1. (Precompactness of the flow implies uniform localization) *Let u be a solution of (1.1) such that*

$$K = \{u(t) : t \in [0, +\infty)\}$$

is precompact in $H^1(\mathbb{R}^3)$. Then for each $\varepsilon > 0$, there exists $R > 0$ so that

$$\int_{|x|>R} |\nabla u(t, x)|^2 dx \leq \varepsilon, \text{ for all } 0 \leq t < +\infty. \quad (7.1)$$

Proof. The proof follows the same steps as in Holmer-Roudenko [17, Lemma 5.6]. So we omit the details \square

We will also need the following local virial identity.

Proposition 7.2. (Virial identity) *Let $\phi \in C_0^\infty(\mathbb{R}^3)$, $\phi \geq 0$ and $T > 0$. For $R > 0$ and $t \in [0, T]$ define*

$$z_R(t) = \int_{\mathbb{R}^3} R^2 \phi\left(\frac{x}{R}\right) |u(t, x)|^2 dx,$$

where u is a solution of (1.1). Then we have

$$z'_R(t) = 2R \operatorname{Im} \int_{\mathbb{R}^3} \nabla \phi\left(\frac{x}{R}\right) \cdot \nabla u \bar{u} dx \quad (7.2)$$

and

$$\begin{aligned} z''_R(t) &= 4 \sum_{j,k} \operatorname{Re} \int \frac{\partial u}{\partial x_k} \frac{\partial \bar{u}}{\partial x_j} \frac{\partial^2 \phi}{\partial x_k \partial x_j} \left(\frac{x}{R}\right) dx - \frac{1}{R^2} \int |u|^2 \Delta^2 \phi\left(\frac{x}{R}\right) dx \\ &\quad - \int |x|^{-b} |u|^4 \Delta \phi\left(\frac{x}{R}\right) dx + R \int \nabla(|x|^{-b}) \cdot \nabla \phi\left(\frac{x}{R}\right) |u|^4 dx. \end{aligned} \quad (7.3)$$

Proof. We first compute z'_R . Note that

$$\partial_t |u|^2 = 2 \operatorname{Re}(u_t \bar{u}) = 2 \operatorname{Im}(i u_t \bar{u}).$$

Since u satisfies (1.1) and using integration by parts, we have

$$\begin{aligned} z'_R(t) &= 2 \operatorname{Im} \int R^2 \phi\left(\frac{x}{R}\right) i u_t \bar{u} dx \\ &= -2 \operatorname{Im} \int R^2 \phi\left(\frac{x}{R}\right) (\Delta u \bar{u} + |x|^{-b} |u|^4) dx \\ &= -2 \operatorname{Im} \int R^2 \phi\left(\frac{x}{R}\right) \nabla \cdot (\nabla u \bar{u}) dx \\ &= 2R \operatorname{Im} \int \nabla \phi\left(\frac{x}{R}\right) \cdot \nabla u \bar{u} dx. \end{aligned}$$

On the other hand, using again integration by parts and the fact that $z - \bar{z} = 2i \operatorname{Im} z$, we obtain

$$\begin{aligned} z''_R(t) &= 2R \operatorname{Im} \int \nabla \phi\left(\frac{x}{R}\right) \cdot (\bar{u}_t \nabla u + \bar{u} \nabla u_t) dx \\ &= 2R \operatorname{Im} \left\{ \sum_j \int \bar{u}_t \partial_{x_j} u \partial_{x_j} \phi\left(\frac{x}{R}\right) dx - u_t \partial_{x_j} \left(\bar{u} \partial_{x_j} \phi\left(\frac{x}{R}\right) \right) dx \right\} \\ &= 2R \operatorname{Im} \left\{ \sum_j 2i \operatorname{Im} \int \bar{u}_t \partial_{x_j} u \partial_{x_j} \phi\left(\frac{x}{R}\right) dx - \int \frac{1}{R} u_t \bar{u} \partial_{x_j}^2 \phi\left(\frac{x}{R}\right) dx \right\} \\ &= 4R I_1 + 2I_2, \end{aligned}$$

where

$$I_1 = Im \sum_j \int \bar{u}_t \partial_{x_j} u \partial_{x_j} \phi \left(\frac{x}{R} \right) \quad \text{and} \quad I_2 = -Im \sum_j \int u_t \bar{u} \partial_{x_j}^2 \phi \left(\frac{x}{R} \right) dx.$$

We start considering I_2 . Since u is a solution of (1.1) we get

$$\begin{aligned} I_2 &= -Im \left\{ \sum_{j,k} \int i \partial_{x_k}^2 u \bar{u} \partial_{x_j}^2 \phi \left(\frac{x}{R} \right) dx \right\} - \sum_j \int |x|^{-b} |u|^4 \partial_{x_j}^2 \phi \left(\frac{x}{R} \right) dx \\ &= Im \left\{ \sum_{j,k} \int i \left(|\partial_{x_k} u|^2 \partial_{x_j}^2 \phi \left(\frac{x}{R} \right) + \frac{1}{R} \partial_{x_k} u \bar{u} \frac{\partial^3 \phi}{\partial x_k \partial x_j^2} \left(\frac{x}{R} \right) \right) dx \right\} \\ &\quad - \int |x|^{-b} |u|^4 \Delta \phi \left(\frac{x}{R} \right) dx \\ &= \int (|\nabla u|^2 - |x|^{-b} |u|^4) \Delta \phi \left(\frac{x}{R} \right) dx + \frac{1}{R} \sum_{j,k} Re \int \partial_{x_k} u \bar{u} \frac{\partial^3 \phi}{\partial x_k \partial x_j^2} \left(\frac{x}{R} \right) dx, \end{aligned}$$

where we have used integration by parts and the fact that $Im(iz) = Re(z)$. Furthermore, since $\partial_{x_k} |u|^2 = 2Re(\partial_{x_k} u \bar{u})$ another integration by parts yields

$$\begin{aligned} I_2 &= \int (|\nabla u|^2 - |x|^{-b} |u|^4) \Delta \phi \left(\frac{x}{R} \right) dx - \frac{1}{2R^2} \sum_{j,k} \int |u|^2 \frac{\partial^4 \phi}{\partial x_k^2 \partial x_j^2} \left(\frac{x}{R} \right) dx \\ &= \int (|\nabla u|^2 - |x|^{-b} |u|^4) \Delta \phi \left(\frac{x}{R} \right) dx - \frac{1}{2R^2} \int |u|^2 \Delta^2 \phi \left(\frac{x}{R} \right) dx. \end{aligned} \quad (7.4)$$

Next, we deduce using the equation (1.1) and $Im(z) = -Im(\bar{z})$ that

$$\begin{aligned} I_1 &= -Im \sum_j u_t \partial_{x_j} \bar{u} \partial_{x_j} \phi \left(\frac{x}{R} \right) dx \\ &= -Imi \sum_j \left\{ \int (\Delta u + |x|^{-b} |u|^2 u) \partial_{x_j} \bar{u} \partial_{x_j} \phi \left(\frac{x}{R} \right) dx \right\} \\ &= -Re \sum_{j,k} \int \partial_{x_k}^2 u \partial_{x_j} \bar{u} \partial_{x_j} \phi \left(\frac{x}{R} \right) dx - \sum_j \int |x|^{-b} \partial_{x_j} \phi \left(\frac{x}{R} \right) |u|^2 Re(\partial_{x_j} \bar{u} u) dx \\ &= -Re \sum_{j,k} \int \partial_{x_k}^2 u \partial_{x_j} \bar{u} \partial_{x_j} \phi \left(\frac{x}{R} \right) dx - \frac{1}{4} \sum_j \int |x|^{-b} \partial_{x_j} \phi \left(\frac{x}{R} \right) \partial_{x_j} (|u|^4) dx \\ &\equiv A + B, \end{aligned}$$

where we have used $Im(iz) = Re(z)$ and $\partial_{x_j} (|u|^4) = 4|u|^2 Re(\partial_{x_j} \bar{u} u)$. Moreover, since $\partial_{x_j} |\partial_{x_k} u|^2 = 2Re \left(\partial_{x_k} u \frac{\partial^2 \bar{u}}{\partial x_k \partial x_j} \right)$ and using integration by parts twice, we get

$$\begin{aligned} A &= Re \sum_{j,k} \left\{ \int \left(\partial_{x_j} \phi \left(\frac{x}{R} \right) \partial_{x_k} u \frac{\partial^2 \bar{u}}{\partial x_k \partial x_j} + \frac{1}{R} \partial_{x_k} u \partial_{x_j} \bar{u} \frac{\partial^2 \phi}{\partial x_j \partial x_k} \left(\frac{x}{R} \right) \right) dx \right\} \\ &= - \sum_{j,k} \frac{1}{2R} \int |\partial_{x_k} u|^2 \partial_{x_j}^2 \phi \left(\frac{x}{R} \right) dx + \frac{1}{R} \sum_{i,j} Re \int \partial_{x_k} u \partial_{x_j} \bar{u} \frac{\partial^2 \phi}{\partial x_j \partial x_k} \left(\frac{x}{R} \right) dx \\ &= - \frac{1}{2R} \int |\nabla u|^2 \Delta \phi \left(\frac{x}{R} \right) dx + \frac{1}{R} \sum_{i,j} Re \int \partial_{x_k} u \partial_{x_j} \bar{u} \frac{\partial^2 \phi}{\partial x_j \partial x_k} \left(\frac{x}{R} \right) dx. \end{aligned}$$

Similarly, integrating by parts

$$\begin{aligned} B &= \frac{1}{4} \sum_j \left(\int \partial_{x_j} \phi \left(\frac{x}{R} \right) \partial_{x_j} (|x|^{-b}) |u|^4 dx + \frac{1}{R} \int \partial_{x_j}^2 \phi \left(\frac{x}{R} \right) |x|^{-b} |u|^4 dx \right) \\ &= \frac{1}{4} \int \nabla \phi \left(\frac{x}{R} \right) \cdot \nabla (|x|^{-b}) |u|^4 dx + \frac{1}{4R} \int \Delta \phi \left(\frac{x}{R} \right) |x|^{-b} |u|^4 dx. \end{aligned}$$

Therefore,

$$\begin{aligned} I_1 &= -\frac{1}{2R} \int |\nabla u|^2 \Delta \phi \left(\frac{x}{R} \right) dx + \frac{1}{R} \sum_{i,j} Re \int \partial_{x_k} u \partial_{x_j} \bar{u} \frac{\partial^2 \phi}{\partial x_j \partial x_k} \left(\frac{x}{R} \right) dx \\ &\quad + \frac{1}{4} \int \nabla \phi \left(\frac{x}{R} \right) \cdot \nabla (|x|^{-b}) |u|^4 dx + \frac{1}{4R} \int \Delta \phi \left(\frac{x}{R} \right) |x|^{-b} |u|^4 dx. \end{aligned} \quad (7.5)$$

Finally it is easy to check that combining (7.4) and (7.5) we obtain (7.3), which complete the proof. \square

Finally, we apply the previous results to proof the rigidity theorem.

Theorem 7.3. (Rigidity) *Let $u_0 \in H^1(\mathbb{R}^3)$ satisfying*

$$E[u_0]^{s_c} M[u_0]^{1-s_c} < E[Q]^{s_c} M[Q]^{1-s_c}$$

and

$$\|\nabla u_0\|_{L^2}^{s_c} \|u_0\|_{L^2}^{1-s_c} < \|\nabla Q\|_{L^2}^{s_c} \|Q\|_{L^2}^{1-s_c}.$$

If the global $H^1(\mathbb{R}^3)$ -solution u with initial data u_0 satisfies

$$K = \{u(t) : t \in [0, +\infty)\} \text{ is precompact in } H^1(\mathbb{R}^3)$$

then u_0 must vanishes, i.e., $u_0 = 0$.

Proof. By Theorem 1.2 we have that u is global in $H^1(\mathbb{R}^3)$ and

$$\|\nabla u(t)\|_{L^2_x}^{s_c} \|u(t)\|_{L^2_x}^{1-s_c} < \|\nabla Q\|_{L^2}^{s_c} \|Q\|_{L^2}^{1-s_c}. \quad (7.6)$$

On the other hand, let $\phi \in C_0^\infty$ be radial, with

$$\phi(x) = \begin{cases} |x|^2 & \text{for } |x| \leq 1 \\ 0 & \text{for } |x| \geq 2. \end{cases}$$

Then, using (7.2), the Hölder inequality and (7.6) we obtain

$$|z'_R(t)| \leq cR \int_{|x| < 2R} |\nabla u(t)| |u(t)| dx \leq cR \|\nabla u(t)\|_{L^2} \|u(t)\|_{L^2} \lesssim cR.$$

Hence,

$$|z'_R(t) - z'_R(0)| \leq |z'_R(t)| + |z'_R(0)| \leq 2cR, \text{ for all } t > 0. \quad (7.7)$$

The idea now is to obtain a lower bound for $z''_R(t)$ strictly greater than zero and reach a contradiction. Indeed, from the local virial identity (7.3)

$$\begin{aligned} z''_R(t) &= 4 \sum_{j,k} Re \int \partial_{x_k} u \partial_{x_j} \bar{u} \frac{\partial^2 \phi}{\partial x_k \partial x_j} \left(\frac{x}{R} \right) dx - \frac{1}{R^2} \int |u|^2 \Delta^2 \phi \left(\frac{x}{R} \right) dx \\ &\quad - \int |x|^{-b} |u|^4 \Delta \phi \left(\frac{x}{R} \right) dx + R \int \nabla (|x|^{-b}) \cdot \nabla \phi \left(\frac{x}{R} \right) |u|^4 dx \\ &= 8 \|\nabla u\|_{L^2_x}^2 - 2(3+b) \| |x|^{-b} |u|^4 \|_{L^1_x} + R(u(t)), \end{aligned} \quad (7.8)$$

where

$$\begin{aligned} R(u(t)) &= 4 \sum_j Re \int \left(\partial_{x_j}^2 \phi \left(\frac{x}{R} \right) - 2 \right) |\partial_{x_j} u|^2 + 4 \sum_{j \neq k} Re \int \frac{\partial^2 \phi}{\partial x_k \partial x_j} \left(\frac{x}{R} \right) \partial_{x_k} u \partial_{x_j} \bar{u} \\ &\quad - \frac{1}{R^2} \int |u|^2 \Delta^2 \phi \left(\frac{x}{R} \right) + R \int \nabla(|x|^{-b}) \cdot \nabla \phi \left(\frac{x}{R} \right) |u|^4 \\ &\quad + \int \left(- \left(\Delta \phi \left(\frac{x}{R} \right) - 6 \right) + 2b \right) |x|^{-b} |u|^4. \end{aligned}$$

Since $\phi(x)$ is radial and $\phi(x) = |x|^2$ if $|x| \leq 1$, the sum of all terms in the definition of $R(u(t))$ integrating over $|x| \leq R$ is zero. Indeed, for the first three terms this is clear by the definition of $\phi(x)$. In the fourth term we have

$$2 \int_{|x| \leq R} \nabla(|x|^{-b}) \cdot x |u|^4 dx = 2 \int_{|x| \leq R} -b |x|^{-b} |u|^4 dx,$$

and adding the last term (also integrating over $|x| \leq R$) we get zero since $\Delta \phi = 6$, if $|x| \leq R$. Therefore, for the integration on the region $|x| > R$, we have the following bound

$$\begin{aligned} |R(u(t))| &\leq c \int_{|x| > R} \left(|\nabla u(t)|^2 + \frac{1}{R^2} |u(t)|^2 + |x|^{-b} |u(t)|^4 \right) dx \\ &\leq c \int_{|x| > R} \left(|\nabla u(t)|^2 + \frac{1}{R^2} |u(t)|^2 + \frac{1}{R^b} |u(t)|^4 \right) dx, \end{aligned} \quad (7.9)$$

where we have used that all derivatives of ϕ are bounded and $|R \partial_{x_j}(|x|^{-b})| \leq c |x|^{-b}$ if $|x| > R$.

Next we use that K is precompact in $H^1(\mathbb{R}^3)$. By Proposition 7.1, given $\varepsilon > 0$ there exists $R_1 > 0$ such that $\int_{|x| > R_1} |\nabla u(t)|^2 \leq \varepsilon$. Furthermore, by Mass conservation (1.2), there exists $R_2 > 0$ such that $\frac{1}{R_2^2} \int_{|x| > R_2} |u(t)|^2 \leq \varepsilon$. Finally, by the Sobolev embedding $H^1 \hookrightarrow L^4$, there exists R_3 such that $\frac{1}{R_3^b} \int_{|x| > R_3} |u(t)|^4 \leq c\varepsilon$ (recall that $\|u(t)\|_{H_x^1}$ is uniformly bounded for all $t > 0$ by (7.6) and Mass conservation (1.2)). Taking $R = \max\{R_1, R_2, R_3\}$ the inequality (7.9) implies

$$|R(u(t))| \leq c \int_{|x| > R} \left(|\nabla u(t)|^2 + \frac{1}{R^2} |u(t)|^2 + \frac{1}{R^b} |u(t)|^4 \right) dx \leq c\varepsilon. \quad (7.10)$$

On the other hand, Lemma 5.1 (iii), (7.8) and (7.10) yield

$$z_R''(t) \geq 16AE[u] - |R(u(t))| \geq 16AE[u] - c\varepsilon,$$

where $A = 1 - w$ and $w = \frac{E[v]^{sc} M[v]^{1-sc}}{E[Q]^{sc} M[Q]^{1-sc}}$.

Now, choosing $\varepsilon = \frac{8A}{c} E[u]$, with c as in (7.10) we have

$$z_R''(t) \geq 8AE[u].$$

Thus, integrating the last inequality from 0 to t we deduce

$$z_R'(t) - z_R'(0) \geq 8AE[u]t. \quad (7.11)$$

Now sending $t \rightarrow \infty$ the left hand of (7.11) also goes to $+\infty$, however from (7.7) it must be bounded. Therefore, we have a contradiction unless $E[u] = 0$ which implies $u \equiv 0$ by Lemma 5.1 (i). \square

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